Unit -3 Matrices -1

➢ Definition: Matrix:

Suppose that *mn* elements a_{ij}, i= 1, 2, 3, …, m, j= 1,2, 3, …, n is given. A rectangular arrangement of these elements in m rows and n columns can be described as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The arrangement of this type is called a matrix of the type m × n or an m × n matrix.

Matrices are denoted by capital letters. It is also denoted by $A_{m \times n}$. 0r $[a_{ij}]_{m \times n}$

e.g. $A=\begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 0 & 7 \end{bmatrix}_{3\times 2}$, $A=\begin{bmatrix} 2 & -4 & -1 \\ 0 & \frac{1}{2} & \frac{1}{4} \\ 1 & 6 & -\frac{7}{4} \end{bmatrix}_{3\times 3}$, $A=\begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}_{2\times 3}$

Note: -

- If all the elements of a matrix are real numbers, then it is called a real matrix and if the elements of a matrix are complex numbers then it is called a complex matrix.

  e.g. $A=\begin{bmatrix} 2 & -4 & -1 \\ 0 & \frac{1}{2} & \frac{1}{4} \\ 1 & 6 & -\frac{7}{4} \end{bmatrix}_{3\times 3}$ is the real matrix and

  $B=\begin{bmatrix} 2 & -4 & -1 \\ 0 & 2+i & 3-i \\ 1 & 6 & i \end{bmatrix}_{3\times 3}$ is complex matrix

- If m = n then matrix is called a square matrix of order n.

e.g. A= $\begin{bmatrix} 2 & -4 & -1 \\ 0 & \frac{1}{2} & \frac{1}{4} \\ 1 & \frac{-5}{2} & -\frac{7}{4} \end{bmatrix}_{3\times 3}$

- A square matrix A of order n is denoted by the symbol = $[a_{ij}]_n$
- A matrix $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ of the type n × 1 is called a column vector or column matrix.
- A matrix $[a_1 \quad a_2 \quad \cdots \quad a_n]$ of the type 1 × n is called a row vector or row matrix.
- If all the elements of an m × n matrix are zeros, then it is called a zero matrix or a null matrix of the type m × n. Zero matrix denoted by simple O.

  e.g. O = $\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$

➢ **Definition: Equality of Matrices: -**

If A =$[a_{ij}]_{m \times n}$ **and B =** $[b_{ij}]_{m \times n}$ be two matrices. and $a_{ij} = b_{ij}$ for *i*= 1, 2, 3, …, m, *j*= 1,2, 3, …, n , then we say that A and B are equal matrices and denoted by A = B.

e.g. (1)A= $\begin{bmatrix} 2 & -4 & -1 \\ 0 & 3 & 4 \\ 1 & 6 & 2 \end{bmatrix}_{3\times 3}$ and B= $\begin{bmatrix} 2 & -4 & -1 \\ 0 & 3 & 4 \\ 1 & 6 & 2 \end{bmatrix}_{3\times 3}$ then A = B

But if A= $\begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 0 & 7 \end{bmatrix}_{3\times 2}$ and B= $\begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 7 \end{bmatrix}_{2\times 3}$ then A ≠B.

➢ **Definition: Addition of Matrices: -**

If A =$[a_{ij}]_{m \times n}$ and B = $[b_{ij}]_{m \times n}$ be two matrices. Their addition A +B is defined by the matrix C = $[c_{ij}]_{m \times n}$ where $c_{ij} = a_{ij} + b_{ij}$ for *i*= 1, 2, 3, …, m, *j*= 1,2, 3, …, n .

e.g.(1) A= $\begin{bmatrix} 2 & -4 & -1 \\ 0 & 3 & 4 \\ 1 & 6 & 2 \end{bmatrix}_{3\times3}$ and B= $\begin{bmatrix} 3 & 3 & -1 \\ 0 & 1 & -3 \\ -1 & 3 & 2 \end{bmatrix}_{3\times3}$ then

A +B = $\begin{bmatrix} 5 & -1 & -2 \\ 0 & 4 & 1 \\ 0 & 9 & 4 \end{bmatrix}_{3\times3}$

also, we can find

A-B= $\begin{bmatrix} -1 & -7 & 0 \\ 0 & 2 & 7 \\ 2 & 3 & 0 \end{bmatrix}_{3\times3}$

- **Note:**

We cannot add two matrices of different orders.

e.g. A= $\begin{bmatrix} 2 & -4 & -1 \\ 0 & 3 & 4 \\ 1 & 6 & 2 \end{bmatrix}_{3\times3}$ and B= $\begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 7 \end{bmatrix}_{2\times3}$

then we cannot find A +B or A-B.

- **Definition: Scalar multiple of Matrices: -**

Let A be an m × n matrix and α be any scalar. Also, A =$[a_{ij}]_{m\times n}$. The scalar multiple "αA" of A is defined by the matrix C = $[c_{ij}]_{m\times n}$ where $c_{ij} = \alpha a_{ij}$ for *i*= 1, 2, 3, …, m, *j*= 1,2, 3, …, n .

e.g. A= $\begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 7 \end{bmatrix}_{2\times3}$ if we take α =3 then αA = 3A = $\begin{bmatrix} 3 & 6 & 0 \\ 3 & -3 & 21 \end{bmatrix}_{2\times3} = C$

- **Definition: Diagonal Matrix: -**

A square matrix whose all the elements except the diagonal elements are zero is called a diagonal matrix.

A = $\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$ is the diagonal matrix.

The elements $a_{11}, a_{22}, \cdots, a_{nn}$ of a square matrix A $=[a_{ij}]_n$ are called diagonal elements.

The diagonal matrix is also written as $diag(a_{11}, a_{22}, \dots, a_{nn})$

➢ **Definition: Identity Matrix: -**

If the diagonal element of diagonal matrix of order n are 1. Then this matrix is called an identity matrix of order n and it is denoted by I_n or I.

I $= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$ is the identity matrix of order n.

I $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the identity matrix of order 3.

I $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ is not an identity matrix but it is diagonal matrix of order 3.

It is also written as $diag(1,3,-1)$

➢ **Definition: Upper triangular Matrix: -**

A square matrix A $=[a_{ij}]_n$, if $a_{ij} = 0$ for $i > j$ then A is called an upper triangular matrix.

ie A $= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$ is upper triangular matrix of order n.

e.g. A $= \begin{bmatrix} 1 & 1 & 9 \\ 0 & 3 & 2 \\ 0 & 0 & -1 \end{bmatrix}$ is an upper triangular matrix of order 3.

➢ **Definition: Lower triangular Matrix: -**

A square matrix A $=[a_{ij}]_n$, if $a_{ij} = 0$ for $i < j$ then A is called lower triangular matrix.

ie A = $\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ is lower triangular matrix of order n.

e.g. A = $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 5 & 4 & -1 \end{bmatrix}$ is lower triangular matrix of order 3.

- **Properties of Matrix:**

Let A, B and C are square matrices and α and β are any scalar. Then,

(1) A + B = B + A (Commutativity)
(2) A + (B + C) = (A + B) + C (Associativity)
(3) α (A + B) = αA + αB,
(4) (α+β) A = α A+βA
(5) (αβ) A = α (βA)

- **Example:** If A = $\begin{bmatrix} 1 & 0 & 3 \\ 1 & 3 & 9 \\ 5 & 4 & -1 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 2 \\ 5 & 4 & -1 \end{bmatrix}$ then prove that commutative law of addition is satisfied . or prove that A + B = B + A.

Solution: Since A + B = $\begin{bmatrix} 1 & 0 & 3 \\ 1 & 3 & 9 \\ 5 & 4 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 2 \\ 5 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 6 & 11 \\ 10 & 8 & -2 \end{bmatrix}$

And B + A= $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 2 \\ 5 & 4 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 3 \\ 1 & 3 & 9 \\ 5 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 6 & 11 \\ 10 & 8 & -2 \end{bmatrix}$

Thus, we get, A + B = B + A

∴ the commutative law of addition is satisfied.

- **Definition: Multiplication of the Matrices: -**

Let A =$[a_{ij}]$ and B = $[b_{jk}]$ be matrices of the types m × n and n × p respectively. Their product AB is defined by the matrix C = $[c_{ik}]$

where $c_{ik} = \sum_{k=1}^{n} a_{ij} b_{jk}$ = (ith row vector of A) (kth column vector of B), i= 1,2,3…,m and k = 1,2,…,p

clearly AB is of the type m × p

- **Note:** If the number of columns of A is equal to the number of rows of B, then and only then AB is defined. Sometimes AB is defined but BA may not be defined. If A and B are square matrices of order n, then AB and BA both are defined. But AB may or may not be same as BA.

  e.g.

  (1) A = $\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$ and B == $\begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix}$ then

  AB =$\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix}$= $\begin{bmatrix} -1 & 1 \\ 5 & 10 \end{bmatrix}$

  And BA =$\begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$ = $\begin{bmatrix} 0 & 3 \\ 5 & 9 \end{bmatrix}$

  Thus, we get AB ≠ BA.

  (2) A = $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and B = $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ then

  AB =$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$= $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Thus, we get AB = 0 but A ≠0 and B≠0.

  (3) If A = $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ -1 & 4 & 3 \end{bmatrix}$ B = $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 2 & 2 & 2 \end{bmatrix}$, C = $\begin{bmatrix} 2 & 3 & 4 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

  Then AB = $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ -1 & 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 2 & 2 & 2 \end{bmatrix}$ = $\begin{bmatrix} 9 & 10 & 7 \\ 6 & 7 & 6 \\ 9 & 8 & -1 \end{bmatrix}$

  And AC = $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ -1 & 4 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ = $\begin{bmatrix} 9 & 10 & 7 \\ 6 & 7 & 6 \\ 9 & 8 & -1 \end{bmatrix}$

Thus, we get AB = AC but B≠C

This prove that AB = AC $\not\Rightarrow$B = C

(4) If A = $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ -1 & 4 & 3 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \end{bmatrix}$ then we cannot find AB

But BA = $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ -1 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 16 & 16 \\ 3 & -1 & 2 \end{bmatrix}$

Here B is 2×3 type matrix and A is 3×3 type matrix thus, we get BA is 2×3 type matrix.

- **Note**: If A is a square matrix then AA is denoted by A^2, AAA by A^3 and AAA…A (n times) by A^n.

- **Definition: Transpose of a Matrices: -**

  Let A be an n ×m matrix and A =$[a_{ij}]_{m\times n}$. Then matrix obtained by interchanging rows and columns of A is called transpose of A and is denoted by A^T or A'. Obvious A^T is of the type m × n.

  Thus, we get A =$[a_{ij}]_{m\times n}$ then we get A^T= $[a_{ji}]_{n\times m}$

  e.g.(1) If A = $\begin{bmatrix} 0 & 6 & 5 \\ 3 & 2 & 0 \end{bmatrix}_{2\times 3}$ then A^T= $\begin{bmatrix} 0 & 3 \\ 6 & 2 \\ 5 & 0 \end{bmatrix}_{3\times 2}$

  (2) If I = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then I^T=$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ thus we get I = I^T.

  (3) If O = $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ then O^T=$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ thus we get O = O^T.

- **Statement:** if A and B are m × n matrices then prove that following

  (1) $(A+B)^T = A^T + B^T$

  (2) $(\alpha A)^T = \alpha A^T$

  (3) $(A^T)^T = A$

Proof:

(1) we know that if A and B are $m \times n$ matrices then $(A + B)$ is $m \times n$ matrix

Thus, we get A^T and B^T are $n \times m$ matrices.

$\therefore$ $A^T + B^T$ is $n \times m$ matrix and $(A+ B)^T$ is $n \times m$ matrix.

$\therefore$ we get $(A+ B)^T = A^T + B^T$

(2) we know that if A is $m \times n$ matrix then $(\alpha A)$is $m \times n$ matrix

Thus, we get A^T is $n \times m$ matrix.

$\therefore$ αA^T is $n \times m$ matrix

$\therefore$ $(\alpha A)^T$ is $n \times m$ matrix.

$\therefore$ we get, $(\alpha A)^T = \alpha A^T$.

(3) if A is $m \times n$ matrix

Then, we get A^T is $n \times m$ matrix.

$\therefore$ $(A^T)^T$ is $m \times n$ matrix

$\therefore$ we get, $(A^T)^T = A$.

e.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 2 & 2 & 2 \end{bmatrix}$, B = $\begin{bmatrix} 2 & 3 & 4 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ then . $A^T = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \\ 3 & -1 & 2 \end{bmatrix}$,

$$B^T = \begin{bmatrix} 2 & 2 & 1 \\ 3 & 2 & 1 \\ 4 & 0 & 1 \end{bmatrix}$$

Now, $A + B = \begin{bmatrix} 3 & 5 & 7 \\ 3 & 3 & -1 \\ 3 & 3 & 3 \end{bmatrix}$

Therefore, $(A + B)^T = \begin{bmatrix} 3 & 3 & 3 \\ 5 & 3 & 3 \\ 7 & -1 & 3 \end{bmatrix}$ and

$$A^T + B^T = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \\ 3 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 2 & 1 \\ 3 & 2 & 1 \\ 4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 5 & 3 & 3 \\ 7 & -1 & 3 \end{bmatrix}$$

Thus, we get $(A+B)^T = A^T + B^T$

Theorem: if A and B be matrices of the order m × n and n × p respectively then prove that $(A\,B)^T = B^T A^T$.

Proof: Here A and B be matrices of the order m × n and n × p respectively

∴ A^T is of the order n × m matrix and B^T is of the order p × n matrix.

i.e. Let A $=[a_{ij}]_{m\times n}$ then we get $A^T = [a_{ji}]_{n\times m}$

and if B $=[b_{jk}]_{n\times p}$ then we get $B^T = [b_{kj}]_{p\times n}$

Here AB is defined, and it is an m × p matrix.

i.e. AB $=[a_{ij}]_{m\times n}[b_{jk}]_{n\times p} = [c_{ik}]_{m\times p}$ say

∴ $(A\,B)^T = [c_{ki}]_{p\times m}$ ---------------------(1)

Now,

$B^T A^T = [b_{kj}]_{p\times n}[a_{ji}]_{n\times m} = [c_{ki}]_{p\times m}$ Say

∴ $B^T A^T = [c_{ki}]_{p\times m}$ -------------------------(2)

From (1) and (2)

we get $(A\,B)^T = B^T A^T$.

(OR)

The (i, k) th element of $B^T A^T = \sum_{k=1}^{n} b_{kj}a_{ji}$ = The (k, i) th element of AB = The (i, k) th element of $(AB)^T$.

Thus, we get $(A\,B)^T = B^T A^T$

- **Definition: Symmetric and skew-symmetric Matrices: -**

Let A $=[a_{ij}]_n$ be a square matrix of order n. If $A = A^T$ then A is called a symmetric matrix and if A If $A = -A^T$ then A is called a skew-symmetric matrix.

e.g. (1) A = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 3 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & -1 \\ 2 & 2 & 6 & 5 \\ 1 & -1 & 5 & 0 \end{bmatrix}$ are symmetric matrices.

(1) A = $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ and B = $\begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & -1 \\ -2 & -2 & 0 & -5 \\ -1 & 1 & 5 & 0 \end{bmatrix}$ are skew-symmetric matrices.

➢ **Note**:

- The diagonal elements of a skew symmetric matrix are zero.
- Any square matrix A =$[a_{ij}]_n$ can be expressed as a sum of a symmetric and skew symmetric matrices.

  A = $\boldsymbol{\frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)}$**.**
- If A is a symmetric matrix then $a_{ij} = a_{ji}$

➢ **Example:** Express the matrix A =$\begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix}$ as a sum of symmetric and skew-symmetric matrix.

Solution: We know that Any square matrix A =$[a_{ij}]_n$ can be expressed as a sum of a symmetric and skew symmetric matrices as

A = $\frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$.

Here A =$\begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix}$ and $A^T = \begin{bmatrix} -1 & 2 & 5 \\ 7 & 3 & 0 \\ 1 & 4 & 5 \end{bmatrix}$

Thus $(A + A^T) = \begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 5 \\ 7 & 3 & 0 \\ 1 & 4 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 9 & 6 \\ 9 & 6 & 4 \\ 6 & 4 & 10 \end{bmatrix}$

And $(A - A^T) = \begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix} - \begin{bmatrix} -1 & 2 & 5 \\ 7 & 3 & 0 \\ 1 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 5 & -4 \\ -5 & 0 & 4 \\ 4 & -4 & 0 \end{bmatrix}$

∴ R.H.S. =$\frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$. =$\frac{1}{2}\begin{bmatrix} -2 & 9 & 6 \\ 9 & 6 & 4 \\ 6 & 4 & 10 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 5 & -4 \\ -5 & 0 & 4 \\ 4 & -4 & 0 \end{bmatrix}$.

$$= \frac{1}{2}\begin{bmatrix} -2 & 14 & 2 \\ 4 & 6 & 8 \\ 10 & 0 & 10 \end{bmatrix} = \begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix} = A\text{=L.H.S.}$$

Thus, we get L.H.S.=R.H.S.

- **Definition: conjugate Matrices: -**
  Let A be m × n complex matrix and A $=[a_{ij}]_{m\times n}$. A matrix obtained by replacing a_{ij} by $[\overline{a_{ij}}]$ is called the conjugate matrix of A and is denoted by $\bar{A}$. The transpose of $\bar{A}$ i.e. $(\bar{A})^T$ is called the transposed conjugate of A. It is denoted by A*. Thus (A*) = $(\bar{A})^T$. Obviously $(\bar{A})^T = \overline{(A^T)}$.

  e.g. A= $\begin{bmatrix} 1+i & -2i \\ 3-i & 4+i \\ 5 & 3-2i \end{bmatrix}_{3\times 2}$

  $\therefore \bar{A} = \begin{bmatrix} 1-i & 2i \\ 3+i & 4-i \\ 5 & 3+2i \end{bmatrix}_{3\times 2}$

  $\therefore$(A*) = $(\bar{A})^T = \begin{bmatrix} 1-i & 3+i & 5 \\ 2i & 4-i & 3+2i \end{bmatrix}_{2\times 3}$

- **Definition: Hermitian and skew-Hermitian Matrices: -**
  Let A $=[a_{ij}]_n$ be a square matrix of order n. If A = A^* then A is called a Hermitian matrix and If A = - A^* then A is called a skew-Hermitian matrix.

e.g. (1) Let A = $\begin{bmatrix} 1+i & -2i \\ 2i & 4+i \end{bmatrix}_{2\times 2}$

$\therefore \bar{A} = \begin{bmatrix} 1-i & 2i \\ -2i & 4-i \end{bmatrix}_{2\times 2}$

$\therefore$(A*) = $(\bar{A})^T = \begin{bmatrix} 1-i & -2i \\ 2i & 4-i \end{bmatrix}_{2\times 2} \neq$ A 0r -A

Therefore, it is neither Hermitian nor skew Hermitian matrix.

(2) Let A = $\begin{bmatrix} 1 & 2+i \\ 2-i & 4 \end{bmatrix}_{2\times 2}$

$\therefore \bar{A} = \begin{bmatrix} 1 & 2-i \\ 2+i & 4 \end{bmatrix}_{2\times 2}$

$\therefore (A^*) = (\bar{A})^T = \begin{bmatrix} 1 & 2+i \\ 2-i & 4 \end{bmatrix}_{2\times 2} = A$

Therefore, it is Hermitian.

(3) Let A = $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}_{2\times 2}$

$\therefore \bar{A} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}_{2\times 2}$

$\therefore (A^*) = (\bar{A})^T = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}_{2\times 2} = - \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}_{2\times 2}$ -A

Therefore, it is skew -Hermitian matrix.

(3) Let A = $\begin{bmatrix} i & 1+i \\ 2-i & i \end{bmatrix}_{2\times 2}$

$\therefore \bar{A} = \begin{bmatrix} -i & 1-i \\ 2+i & -i \end{bmatrix}_{2\times 2}$

$\therefore (A^*) = (\bar{A})^T = \begin{bmatrix} -i & 2+i \\ 1-i & -i \end{bmatrix}_{2\times 2} \neq A$ 0r -A

Therefore, it is neither Hermitian nor skew Hermitian matrix.

- **Note**:
  - In a Hermitian matrix all diagonal elements are real numbers.
  - The diagonal elements of a skew Hermitian matrix are either zero or purely

complex numbers.

➢ **Prove that**

(i) (A+ B)* = A* + B* (ii) (A*)* = A (iii) (AB)* =B*A* (iv) (αA)* = $\overline{\alpha}$A*

Proof:- (i) Let A =$[a_{ij}]_n$ and A =$[b_{ij}]_n$ be a complex square matrices of order n.

Here i= j

. ∴ (A + B) = ($[a_{ii}]_n$ +$[b_{ii}]_n$) = $[c_{ii}]_n$

. ∴ (A + B)* = $(\overline{A+B}\ )^T = (\bar{A}+\bar{B})^T \quad = [\overline{c_{ii}}\ ]_n{}^T$=($[\overline{a_{ii}}]^T{}_n$ +$[\overline{b_{ii}}]^T{}_n$)

= A* + B*

∴ (A+ B)* = A* + B*

(ii) (A*)* =$(\overline{[\overline{a_{ii}}]^T{}_n})^T$ =($[a_{ii}]_n$ =A

∴ (A*)* = A

(iii) (AB)* == $(\overline{AB}\ )^T = [\overline{c_{ii}}\ ]_n{}^T = [\overline{c_{ii}}\ ]_n$

Now B*A* = $[\overline{b_{ii}}]^T{}_n$ ($[\overline{a_{ii}}]^T{}_n$) =$[\overline{b_{ii}}][\overline{a_{ii}}]$=$[\overline{c_{ii}}\ ]_n$

Example: - If A is square matrix, then prove that A +A* is Hermitian and A-A* is skew Hermitian.

Solution: - Here (A +A*)* = A* +(A*)* = A* +A
Thus, we prove that (A +A*)* = A+A*
(∵ definition of Hermitian matrix as A* = A)
∴A +A* is Hermitian matrix.
Now, (A -A*)* = A* -(A*)* = A* -A
Thus, we prove that (A -A*)* = -A+A* = -(A -A*)
(∵ definition of Skew Hermitian matrix as A* =- A)
∴A -A* is skew Hermitian matrix.

➢ **Example: -** If A and B are symmetric matrices of the same order, then prove that AB – BA is a skew- symmetric matrix.

Solution: - Here A and B are symmetric matrices is given.

$\therefore$ $A = A^T$ and $B = B^T$.

Now, $(AB - BA)^T = (AB)^T - (BA)^T = B^T A^T - A^T B^T = BA - AB = -(AB - BA)$

Thus, we get $(AB - BA)^T = -(AB - BA)$

$\therefore$ this prove that AB – BA is a skew- symmetric matrix.

- **Example: -** If A and B are Hermitian matrices of the same order, then prove that AB – BA is a skew- Hermitian matrix.

Solution: - Here A and B are Hermitian matrices is given.

$\therefore$ $A = A^*$ and $B = B^*$.

Now, $(AB - BA)^* = (AB)^* - (BA)^* = B^* A^* - A^* B^* = BA - AB = -(AB - BA)$

Thus, we get $(AB - BA)^* = -(AB - BA)$

$\therefore$ this prove that AB – BA is a skew- Hermitian matrix.

- **Example: -** If A is skew-Hermitian matrix, then prove that *i*A is a Hermitian matrix.

Solution: - Here A is skew-Hermitian matrix is given.

$\therefore$ $A^* = -A$.

Now, $(iA)^* = \bar{\imath}A^* = -i(-A) = iA$

Thus, we get $(iA)^* = iA$

$\therefore$ this prove that *i*A is a Hermitian matrix.

- **Example: -** If A is Hermitian matrix, then prove that *i*A is a skew-Hermitian matrix.

Solution: - Here A is Hermitian matrix is given.

$\therefore$ $A^* = A$.

Now, $(iA)^* = \bar{\imath}A^* = -i(A) = -iA$

Thus, we get $(iA)^* = -\,iA$

$\therefore$ this prove that iA is a skew-Hermitian matrix.

Elementary Operation:

- **Definition: Elementary row operations:**

  The following operations for the matrix A are called elementary row operations:

  (1) Interchange of two rows

  (2) The multiplication of i^{th} row by a non-zero number k. $r_i(k)$.

  (3) Adding to i^{th} row the multiplication by k of the j^{th} row. $(r_i + kr_j)$

  e.g(1) $A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & -1 \\ 2 & 2 & 6 & 5 \\ 1 & -1 & 5 & 0 \end{bmatrix}$

  Interchange of two rows of the given matrix A i.e. Interchange of r_1 and r_2.

  Then, we get the matrix $B = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 1 & 0 & 2 & 1 \\ 2 & 2 & 6 & 5 \\ 1 & -1 & 5 & 0 \end{bmatrix}$ Here A∼B.

  Now, The multiplication of 3^{th} row by a non-zero number 2. $R_3(2)$.

  Then we get the matrix B as $C = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 1 & 0 & 2 & 1 \\ 4 & 4 & 12 & 10 \\ 1 & -1 & 5 & 0 \end{bmatrix}$ Here B∼C.

  (4) Now, Adding to 4^{th} row the multiplication by -3 of the 2^{th} row. $(r_4 + (-3)r_2)$

  $D = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 1 & 0 & 2 & 1 \\ 4 & 4 & 12 & 10 \\ -2 & -1 & -1 & -3 \end{bmatrix}$ Here C∼D.

  Thus, we get A∼D

- **Definition: Elementary column operations:**

  The following operations for the matrix A are called elementary column operations:

  (1) Interchange of two column.

  (2) The multiplication of i^{th} column by a non-zero number k. $c_i(k)$.

  (3) Adding to i^{th} column the multiplication by k of the j^{th} column. $(c_i + kc_j)$

- **Definition: Row equivalent matrix:**
  The matrix B obtained by performing a finite numbers of elementary operations finite numbers of times to a matrix A is called row equivalent matrix to A.
  It is denoted by B∼A.

- **Definition: inverse of the matrix:**
  If A and B are square matrices of same type such that AB = BA = I, then B is called the inverse of A. Here I is the identity matrix.
  The inverse of A is denoted by A^{-1}.
  e.g. A= $\begin{bmatrix} 1 & 2 & 4 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and B= $\begin{bmatrix} 1 & -2 & -2 \\ 2 & -3 & -5 \\ -1 & 2 & 2 \end{bmatrix}$
  Then AB = BA = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ = I
  Therefore, B is the inverse of A. i.e. B = A^{-1}.
- Note:
  - The determinant of a square matrix A is denoted by detA or $|A|$.

  i.e. $|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}$

- **Definition: minor (or sub determinant)**
  The minor (or sub determinant) of any element of a determinant is the determinant obtained by the row and column in which that element occurs.

e.g. the minor of b_2 in $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ i.e $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$

Similarly, the minor of b_3 in $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ i.e $\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

And the minor of a_1 in $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is i.e $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$

- **Definition: Cofactor: -**
  The value of minor sign i. e. the coefficient of an element in the expansion of determinant is called the cofactor of this element.

i.e. In the matrix $[a_{ij}]_n$ The cofactor of $a_{ij} = (-1)^{i+j}$minor of a_{ij}

The cofactor of a_{ij} is denoted by the symbol A_{ij}.

Thus, A_{ij} = cofactor of $a_{ij} = (-1)^{i+j}$minor of a_{ij}

i.e. **e.g.** the minor of b_2 in $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ i.e $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$

therefore, the cofactor of b_2 in $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (-1)^{2+2} \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} = A_{22}$

Similarly, the minor of b_3 in $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ i.e $\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

therefore, the cofactor of b_3 in $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (-1)^{2+3} \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = A_{32}$

And the minor of a_1 in $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is i.e $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$

therefore, the cofactor of a_1 in $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (-1)^{1+1} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = A_{11}$

- **Definition: - Adjoint Matrix: -**

  Let A = $[a_{ij}]_n$ be square matrix of order n. If B = $[b_{ij}]_n$ is a square matrix such that $b_{ij} = A_{ij}$ $(i, j = 1,2,3, \dots, n)$, Then B is called the adjoint matrix of A i.e. $adjA = [A_{ij}]^T$.

  It is denoted by the symbol $adjA$, where A_{ij} is the cofactor of a_{ij}.

$$[A_{ij}] = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1} & a_{n2} & \cdots & A_{nn} \end{bmatrix}$$

- **Example:** Find the $adjA$ for A = $\begin{bmatrix} 2 & 3 & -1 \\ -1 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix}$.

Solution: Here A = $\begin{bmatrix} 2 & 3 & -1 \\ -1 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix}$

$A_{11} = (-1)^{1+1} \begin{bmatrix} 2 & 3 & -1 \\ -1 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix} = (-1)^{1+1} \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} = \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} = 6 - 2 = 4$

Similarly, $A_{12} = (-1)^{1+2} \begin{vmatrix} -1 & 2 \\ 2 & 2 \end{vmatrix} = -(-2-4) = 6$

$A_{13} = (-1)^{1+3} \begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix} = -1- 6 = -7$

$A_{21} = (-1)^{2+1} \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} = -(6+1) = -7$, $A_{22} = (-1)^{2+2} \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} = 4+2 = 6$,

$A_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} = -(2-6) = 4$, $A_{31} = (-1)^{3+1} \begin{vmatrix} 3 & -1 \\ 3 & 2 \end{vmatrix} = (6+3) = 9$,

$A_{32} = (-1)^{3+2} \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = -(4-1) = -3$, $A_{33} = (-1)^{3+3} \begin{vmatrix} 2 & 3 \\ -1 & 3 \end{vmatrix} = (6+3) = 9$,

$$[A_{ij}] = \begin{bmatrix} 4 & 6 & -7 \\ -7 & 6 & 4 \\ 9 & -3 & 9 \end{bmatrix} \text{ thus, } adjA = [A_{ij}]^T = \begin{bmatrix} 4 & -7 & 9 \\ 6 & 6 & -3 \\ -7 & 4 & 9 \end{bmatrix}$$

- **Theorem:** If A = $[a_{ij}]_n$ be square matrix of order n. Then prove that A $(adjA)$ = $(adjA)$ A = $|A|I_n$

Proof: Let B = $adjA$ and C = A $(adjA)$

If B = $[b_{ij}]_n$ and C = $[c_{ij}]_n$, then $c_{ip} = \sum_{j=1}^{n} a_{ij}b_{jp} = \sum_{j=1}^{n} a_{ij}b_{pj}$

$$\therefore c_{ip} = \begin{cases} 0 & if\ i \neq p \\ |A| & if\ i = p \end{cases}$$

(used of a theorem about the expansion of the determinant.)

Therefore, A $(adjA)$ = $|A|I_n$

Similarly, it can be proved that $(adjA)$ A = $|A|I_n$

Thus if $|A| \neq 0$ then $\frac{A.adjA}{|A|} = \frac{adjA.A}{|A|} = I_n$

Therefore, $A^{-1} = \frac{adjA}{|A|}$.

- **Example:** Find $A^{-1}$for the matrix A = $\begin{bmatrix} 2 & 1 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$.

Solution: - Here A = $\begin{bmatrix} 2 & 1 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$

det A = 2(12-2) -1(16-1) +4 (8-3) = 20 – 15 + 20 = 25 $\neq$0

A_{11} = $(-1)^{1+1} \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$ = 10, A_{12} =$(-1)^{1+2} \begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix}$ = -15

A_{13} =$(-1)^{1+3} \begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix}$ = 5, A_{21} =$(-1)^{2+1} \begin{vmatrix} 1 & 4 \\ 2 & 4 \end{vmatrix}$) = 4,

A_{22} =$(-1)^{2+2} \begin{vmatrix} 2 & 4 \\ 1 & 4 \end{vmatrix}$ =4, A_{23} =$(-1)^{2+3} \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$ = -3,

A_{31} =$(-1)^{3+1} \begin{vmatrix} 1 & 4 \\ 3 & 1 \end{vmatrix}$ = -11, A_{32} =$(-1)^{3+2} \begin{vmatrix} 2 & 4 \\ 4 & 1 \end{vmatrix}$ = 14,

A_{33} =$(-1)^{3+3} \begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix}$ = 2

$$\therefore adjA = \begin{bmatrix} 10 & 4 & -11 \\ -15 & 4 & 14 \\ 5 & -3 & 2 \end{bmatrix}$$

Therefore, $A^{-1} = \frac{adjA}{|A|}. = \frac{1}{25}\begin{bmatrix} 10 & 4 & -11 \\ -15 & 4 & 14 \\ 5 & -3 & 2 \end{bmatrix}$

Thus, we get $A^{-1} = \frac{1}{25}\begin{bmatrix} 10 & 4 & -11 \\ -15 & 4 & 14 \\ 5 & -3 & 2 \end{bmatrix}$

Check: - we know that $AA^{-1} = A^{-1}A = I_n$

So, $AA^{-1} = \begin{bmatrix} 2 & 1 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix} . \begin{bmatrix} \frac{10}{25} & \frac{4}{25} & -\frac{11}{25} \\ -\frac{15}{25} & \frac{4}{25} & \frac{14}{25} \\ \frac{5}{25} & -\frac{3}{25} & \frac{2}{25} \end{bmatrix}$

$$= \begin{bmatrix} \frac{20-15+20}{25} & \frac{8+4-12}{25} & \frac{-22+14+8}{25} \\ \frac{40-45+5}{25} & \frac{16+12-3}{25} & \frac{-44+42+2}{25} \\ \frac{10-30+20}{25} & \frac{4+8-12}{25} & \frac{-11+28+8}{25} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Some properties of inverse:

- **Theorem:** If A and B are invertible matrices of order n, then prove that $(AB)^{-1} = B^{-1}A^{-1}$.
  Proof: $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I_n)A^{-1} = AA^{-1} = I_n$
  And $(B^{-1}A^{-1})(AB) = B(A^{-1}A)B^{-1} = B(I_n)B^{-1} = BB^{-1} = I_n$
  Thus, we get $(AB)(B^{-1}A^{-1}) = I_n = (B^{-1}A^{-1})(AB)$
  $\therefore (B^{-1}A^{-1}) = I_n(AB)^{-1} = (AB)^{-1}$.
  This prove that $(B^{-1}A^{-1}) = AB)^{-1}$.

- **Theorem:** If A is an invertible square matrix of order n and B and C are matrices of the order n×p, then prove that AB=AC implies B = C.
- **Proof:** Here A is an invertible square matrix of order n and B and C are matrices of the order n×p is given

Now, AB=AC

$\Rightarrow A^{-1}AB=A^{-1}AC$ multiple A^{-1} on both side

$\Rightarrow (A^{-1}A)B=(A^{-1}A)C$

$\Rightarrow (I_n)B=(I_n)C$

$\Rightarrow B = C$

This prove that AB=AC implies B = C.

- **Theorem:** If A is an invertible square matrix of order n then prove that $(A^T)^{-1} = (A^{-1})^T$.

Proof: Here A is an invertible square matrix of order n is given.

Now,we have $AA^{-1} = A^{-1}A = I$

Taking transposes, we obtain

$(A)^T(A^{-1})^T= (A^{-1})^T(A)^T = I^T = I$

$\Leftrightarrow (A^{-1})^T$ is the inverse of A^T

$\Rightarrow (A)^T(A^{-1})^T= I$

$\Rightarrow (A^T)^{-1} = (A^{-1})^T$.

- **Theorem:** If A is an invertible square matrix of order n then prove that (a) $(A^{-1})^{-1} = A$. (b) $(\bar{A})^{-1} = \overline{(A^T)}$ (c) $(A^*)^{-1} = (A^{-1})^*$

Proof: (a) Now,we have $AA^{-1} = A^{-1}A = I$

Taking inverses, we obtain

$(A)^{-1}(A^{-1})^{-1}= (A^{-1})^{-1}(A)^{-1} = I^{-1} = I$

$\Leftrightarrow (A^{-1})^{-1}$ is the inverse of A^{-1}

$\Rightarrow (A)^{-1}(A^{-1})^{-1} = I$

i.e. If we take $(A^{-1})^{-1}(A)^{-1} = I$ thus we get , $(A^{-1})^{-1}=I/(A)^{-1}$

$\Rightarrow (A^{-1})^{-1} = AI = A$.

Similary you can prove (b) and (c) .

The row and column rank:

Definition:- linear combination:

If α_1, α_2,α_3, ….,α_n are scalars (real or complex) and if C_1, C_2, C_3, ….,C_n are column vectors or column matrices, then $\alpha_1 C_1+\alpha_2 C_2+\alpha_3 C_3+ \dots +\alpha_n C_n$ is called linear combination of C_1, C_2, C_3, ….,C_n.

Similarly, If β_1, β_2,β_3, ….,β_n are scalars (real or complex) and if R_1, R_2, R_3, ….,R_n are Row vectors or Row matrices, then $\beta_1 R_1+\beta_2 R_2+\beta_3 R_3+ \dots +\beta_n R_n$ is called linear combination of R_1, R_2, R_3, ….,R_n.

Definition: - Linearly independent (L.I.): -

A set { u_1, u_2, u_3, ….,u_n} of vectors is said to be linearly independent(L.I.) if

the linear combination $\alpha_1 u_1+\alpha_2 u_2+\alpha_3 u_3+ \dots +\alpha_n u_n = 0$ with all scalars α_1, α_2,α_3, ….,α_n zero. i.e. all of the $\alpha's$is zero.

i.e. $\alpha_1 = 0$, $\alpha_2 = 0$,$\alpha_3 = 0$, ….,$\alpha_n = 0$.

Note: - Here u_1, u_2, u_3, …., u_n may be column vectors or row vectors

Definition: - Linearly dependent (L.D.): -

A set {u_1, u_2, u_3, …., u_n} of vectors is said to be linearly dependent (L.D.) if

the linear combination $\alpha_1 u_1+\alpha_2 u_2+\alpha_3 u_3+ \dots + \alpha_n u_n = 0$ with at least one of

α_1, α_2,α_3, ….,α_n is not zero. i.e. all of the $\alpha's$ is not zero.

Let us suppose that $\alpha_3 \neq 0$

i.e. $\alpha_1 = 0$, $\alpha_2 = 0$,$\alpha_3 \neq 0$, ….,$\alpha_n = 0$.

Note: - Here u_1, u_2, u_3, …., u_n may be column vectors or row vectors.

Example: - Let A = $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ then show that it is linearly independent.

Solution: - Let $C_1=\begin{bmatrix}1\\2\\0\end{bmatrix}$, $C_2=\begin{bmatrix}0\\0\\1\end{bmatrix}$ and $C_3=\begin{bmatrix}1\\0\\1\end{bmatrix}$ be three column vectors Then

we have $\alpha, \beta, \gamma \in R$ such that $\alpha \begin{bmatrix}1\\2\\0\end{bmatrix}+\beta \begin{bmatrix}0\\0\\1\end{bmatrix}+\gamma \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$\alpha + \gamma = 0$, $2\alpha = 0$, $\beta + \gamma = 0$

Thus we get $\alpha = 0$, $\beta = 0$ and $\gamma = 0$

Given column vector are linearly independent.

Let $R_1= (1, 0,1)$, $R_2= (2, 0, 0)$ and $R_3= (0, 1, 1)$ be three row vectors Then

we have $\alpha, \beta, \gamma \in R$ such that $\alpha (1, 0,1)+\beta (2, 0, 0) +\gamma (0, 1, 1) = (0, 0, 0)= 0$

$\alpha + 2\beta = 0$, $\gamma = 0$, $\alpha + \gamma = 0$

Thus we get $\alpha = 0$, $\beta = 0$ and $\gamma = 0$

Given row vector are linearly independent.

Example: - Let A = $\begin{bmatrix}1 & 1 & 1\\1 & 1 & 0\\1 & 0 & 0\end{bmatrix}$ then show that it is linearly independent.

Solution: - Let $C_1=\begin{bmatrix}1\\1\\1\end{bmatrix}$, $C_2=\begin{bmatrix}1\\1\\0\end{bmatrix}$ and $C_3=\begin{bmatrix}1\\0\\0\end{bmatrix}$ be three column vectors Then

we have $\alpha, \beta, \gamma \in R$ such that $\alpha \begin{bmatrix}1\\1\\1\end{bmatrix}+\beta \begin{bmatrix}1\\1\\0\end{bmatrix}+\gamma \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$\alpha + \beta + \gamma = 0$, $\alpha + \beta = 0$, $\alpha = 0$

Thus we get $\alpha = 0$, $\beta = 0$ and $\gamma = 0$

Given column vector are linearly independent.

Let R_1= (1, 1,1), R_2= (1, 1, 0) and R_3= (1, 0, 0) be three row vectors Then

we have $\alpha, \beta, \gamma \in R$ such that α (1, 1,1) +β (1, 1, 0) +γ (1, 0, 0) = (0, 0, 0)

$\alpha + \beta + \gamma = 0, \ \alpha + \beta = 0, \ \alpha = 0$

Thus we get $\alpha = 0$, $\beta = 0$ and $\gamma = 0$

Given row vector are linearly independent.

Example: - Let A = $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}$ then show that it is linearly dependent.

Solution: - Let C_1=$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, C_2= $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$, C_3=$\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$ and C_4=$\begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix}$ be four column vectors

Then

we have $\alpha, \beta, \gamma, \delta \in R$ such that $\alpha \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + \gamma \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} + \delta \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\alpha + 2\beta + 3\gamma + 4\delta = 0, \ 2\alpha + 4\beta + 6\gamma + 8\delta = 0, \ 3\alpha + 6\beta + 9\gamma + 12\delta = 0$

$\alpha + 2\beta + 3\gamma + 4\delta = 0, \ \alpha + 2\beta + 3\gamma + 4\delta = 0, \ \alpha + 2\beta + 3\gamma + 4\delta = 0,$

Thus, we get $\alpha = -2\beta - 3\gamma - 4\delta$

if we take $\beta = 1$ $\gamma = 1$, $\delta = 1$ then we get, $\alpha = -9$

Here $\alpha \neq 0$, $\beta \neq 0$, $\gamma \neq 0$ and $\delta \neq 0$

Given column vector are linearly dependent.

Let R_1= (1, 2,3,4), R_2= (2, 4, 6,8) and R_3= (3, 6, 9,12) be three row vectors Then

we have $\alpha, \beta, \gamma \in R$ such that α (1, 2,3,4) + β (2, 4, 6,8) + γ (3, 6, 9,12) = (0, 0, 0)

$\alpha + 2\beta + 3\gamma = 0, \ 2\alpha + 4\beta + 6\gamma = 0, \ 3\alpha + 6\beta + 9\gamma = 0$ and $4\alpha + 8\beta + 12\gamma = 0$

$\alpha + 2\beta + 3\gamma = 0, \alpha + 2\beta + 3\gamma = 0, \alpha + 2\beta + 3\gamma = 0$ and $\alpha + 2\beta + 3\gamma = 0$

$\alpha = -2\beta - 3\gamma$

if we take β =1 γ =1 then we get, α = -5

Thus, we get $\alpha \neq 0$, $\beta \neq 0$ and $\gamma \neq 0$

Given Row vector are linearly dependent.

- **Definition: - Row rank:**

  The maximum number of linearly independent rows of a matrix is called row rank of the matrix.

- **Definition: - Column rank:**

  The maximum number of linearly independent columns of a matrix is called rank of column he matrix.

- **Definition: echelon form:**

  any matrix A or its row equivalent form satisfies the following conditions then 'A' is said to be in the row reduced echelon form :

(i) The first non-zero entry in non- zero row is 1.
(ii) If a column contains the first non- zero entry of any row. Then every other entry in that column is zero.
(iii) If the matrix has the zero rows (the rows containing only zero) then all zero must occur below all the non-zero rows.
(iv)Let there be r non-zero rows. If the first non-zero entry of the i^{th} non-zero row occurs in the column (i = I, 2, .., r) then $k_1<k_2<k_3<\ldots<k_r$.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 12 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; E = \begin{bmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ are in the row reduced echelon form.

Theorem: If the matrix is in the row reduced echelon form, then the number of non-zero rows is the row rank of the matrix.

Theorem: The row rank of a matrix A is equal to the row rank of row reduced echelon matrix B obtained form A.

Proof: matrix B is obtained from A by applying the row elementary operations on A.

Therefore, it is sufficient to show that operations on the matrix does not after the rank or A. If we interchange any two rows of A, then the number of linearly independent rows A remains the same.

Similarly, the multiplication any row of A by a non-zero number does not alter the number of linearly independent rows of the matrix A.

Let us now consider the third elementary operation. Suppose the row matrix R_1 is multiplied with α and added to the row matrix R_2. Let the remaining row matrices of A are R_3, R_4 ......, R_m.

It can be seen that the sets P = {R_1. R_2, …, R_m} and the set Q= {R_1. $R_2 + \alpha R_1$, …, R_m} are of the same nature i.e. If P is linearly independent, then Q is also linearly independent and if p is linearly dependent then Q is also linearly dependent.

Therefore, the third elementary operation also does not alter the number of independent rows of A. Thus, the theorem is proved.

Note:

Let the matrix A be in the form of row reduced echelon form. Draw horizontal and vertical lines in such away that under these lines and or left-hand sides of these lines only zeros occur. Also, at the point., where the vertical line occurs after the horizontal line only 1 occurs. Such point are called steps.

$$A = \begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{5\times 6}$$ Here A is a 5×6 matrix with four steps.

Theorem : If a matrix is in the row reduced echelon form, then its column rank is equal to the number of steps in the matrix.

Proof : Let the matrix A be in row reduced echelon form with p steps. The column before the first step is a zero column matrix and each row after last step is a zero row matrix.

Therefore, the non-zero column matrices are elements or Rp. Obviously, the column rank of A is P or a natural number was then P. It should be noted that each step gives us a non-zero vector. The ser of these vectors is linearly independent. Hence the column rank of A is or greater than the column rank of A is p.

Example: find the rank of the matrix A = $\begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$

Solution: $r(A) \leq 3$

Let R_1= (2, 3,4,-1), R_2= (5, 2, 0,-1) and R_3= (-4, 5, 12,-1) be three row vectors Then

we have $\alpha, \beta, \gamma \in R$ such that

α (2, 3,4, -1) + β(5, 2, 0, -1) + γ (-4, 5, 12, -1) = (0, 0, 0)

$2\alpha + 5\beta - 4\gamma = 0$, $3\alpha + 2\beta + 5\gamma = 0$, $4\alpha + 12\gamma = 0$ and $-\alpha - \beta - \gamma = 0$

$2\alpha + 5\beta - 4\gamma = 0$,____________________(1)

$3\alpha + 2\beta + 5\gamma = 0$, __________________(2)

$4\alpha + 12\gamma = 0$ __________________(3)

$-\alpha - \beta - \gamma = 0$ ____________________(4)

From equations (1) and (4)

$$\begin{array}{r} 2\alpha + 5\beta - 4\gamma = 0 \\ \underline{-2\alpha - 2\beta - 2\gamma = 0} \\ 3\beta - 6\gamma = 0 \end{array} \quad \Rightarrow \ \beta - 2\gamma = 0 \text{ \_\_\_\_\_\_\_\_\_\_(5)}$$

From equations (2) and (4)

$$\begin{array}{r} 3\alpha + 2\beta + 5\gamma = 0 \\ \underline{-3\alpha - 3\beta - 3\gamma = 0} \\ -\beta + 2\gamma = 0 \end{array} \quad \Rightarrow \ \beta - 2\gamma = 0 \text{ \_\_\_\_\_\_\_\_\_\_(6)}$$

From equations (3) and (4)

$$\begin{array}{r} 4\alpha + 0\beta + 12\gamma = 0 \\ \underline{-4\alpha - 4\beta - 4\gamma = 0} \\ -4\beta + 8\gamma = 0 \end{array} \quad \Rightarrow \ \beta - 2\gamma = 0 \text{ \_\_\_\_\_\_\_\_\_\_(7)}$$

From equations (5), (6) and (7) we get, $\beta - 2\gamma = 0 \Rightarrow \beta = 2\gamma$

Put $\beta = 2\gamma$ in equation (4) then we get $\alpha = \gamma$.

Thus, if we take $\gamma = 1$.then we get $\beta = 2$ and $\alpha = 1$

Thus, we get $\alpha \neq 0$, $\beta \neq 0$ and $\gamma \neq 0$

Given Row vector are linearly dependent.

Now let remove R_3 Row and check the linearly independent or dependent for R_1= (2, 3,4,-1), R_2= (5, 2, 0,-1)

we have $\alpha, \beta \in R$ such that

α (2, 3,4, -1) + β(5, 2, 0, -1) = (0, 0, 0)

$2\alpha + 5\beta = 0$, $3\alpha + 2\beta = 0$, $4\alpha = 0$ and $-\alpha - \beta = 0$

we get, $\alpha = 0$ $\beta = 0$, $\gamma = 0$

Given Row vector R_1= (2, 3,4, -1), R_2= (5, 2, 0, -1) are linearly independent.

Rank of matrix A = r(A) = 2.

(OR)

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$$

Solution: $r(A) \leq 3$

Let $C_1 = \begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}$, $C_2 = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ and $C_3 = \begin{bmatrix} 4 \\ 0 \\ 12 \end{bmatrix}$ be three column vectors Then

we have $\alpha, \beta, \gamma \in R$ such that

$$\alpha \begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix} + \beta \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} + \gamma \begin{bmatrix} 4 \\ 0 \\ 12 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2\alpha\ +3\beta\ +4\gamma = 0,\ 5\alpha+2\beta\ +0\gamma\ = 0,$ and $-4\alpha\ +5\beta\ +12\gamma = 0$

$2\alpha\ +3\beta\ +4\gamma = 0,$ ____________________(1)

$5\alpha+2\beta\ \ = 0,$ __________________(2)

$-4\alpha\ +5\beta\ +12\gamma = 0$ ___________________(3)

From equations (1) and (3)

$$\begin{array}{l} 6\alpha + 9\beta + 12\gamma = 0 \\ \underline{4\alpha - 5\beta - 12\gamma = 0} \\ 10\alpha + 4\beta \qquad\quad = 0 \quad \Rightarrow\ 5\alpha + 2\beta = 0 \ \text{\_\_\_\_\_\_\_\_\_\_\_(4)} \end{array}$$

From equations (2) and (4)

we get $5\alpha + 2\beta = 0\ \Rightarrow \alpha = -\frac{2}{5}\beta$

Put $\alpha = -\frac{2}{5}\beta$ in equation (1) then we get $\gamma = -\frac{11}{20}\beta$.

Thus, if we take $\beta\ = 1$.then we get $\gamma = -\frac{11}{20}$ and $\alpha\ = -\frac{2}{5}$

Thus, we get $\alpha \neq 0$, $\beta \neq 0$ and $\gamma \neq 0$

Given column vectors are linearly dependent.

Now let remove C_3 column and check the linearly independent or dependent for

$C_1=\begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}$, $C_2=\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$

we have $\alpha, \beta \in R$ such that

$$\alpha \begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix} + \beta \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2\alpha\ +3\beta = 0,\ 5\alpha+2\beta = 0$ and $-4\alpha\ +5\beta\ = 0$

we get, $\alpha = 0\ , \beta = 0$

Given column vectors $C_1 = \begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}$, $C_2 = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ are linearly independent.

Rank of matrix A = r(A) = 2.

Example: find the rank of the matrix $A = \begin{bmatrix} 4 & 3 & 0 & -2 \\ 3 & 4 & -1 & -3 \\ -7 & -7 & 1 & 5 \end{bmatrix}$

Solution: $r(A) \leq 3$

Let R_1= (4, 3,0, -2), R_2= (3, 4, -1, -3) and R_3= (-7, -7, 1,5) be three row vectors Then

we have $\alpha, \beta, \gamma \in R$ such that

α (4, 3,0, -2) + β(3, 4, -1, -3) + γ (-7, -7, 1,5) = (0, 0, 0)

$4\alpha + 3\beta - 7\gamma = 0$, $3\alpha + 4\beta - 7\gamma = 0$, $0\alpha - \beta + \gamma = 0$ and $-2\alpha - 3\beta + 5\gamma = 0$

$4\alpha + 3\beta - 7\gamma = 0$ ____________________(1)

$3\alpha + 4\beta - 7\gamma = 0$, __________________(2)

$0\alpha - \beta + \gamma = 0$ ___________________(3)

$-2\alpha - 3\beta + 5\gamma = 0$ ____________________(4)

From equations (1) and (4)

$$\begin{array}{r} 4\alpha + 3\beta - 7\gamma = 0 \\ \underline{-4\alpha - 6\beta + 10\gamma = 0} \\ -3\beta + 3\gamma = 0 \end{array} \quad \Rightarrow \ \beta - \gamma = 0 \ \text{\_\_\_\_\_\_\_\_\_\_\_(5)}$$

From equations (2) and (4)

$$\begin{array}{r} 6\alpha + 8\beta - 14\gamma = 0 \\ \underline{-6\alpha - 9\beta + 15\gamma = 0} \\ -\beta + \gamma = 0 \end{array} \quad \Rightarrow \ \beta - \gamma = 0 \ \text{\_\_\_\_\_\_\_\_\_\_\_(6)}$$

From equations (3)

$\Rightarrow \ \beta - \gamma = 0$ ___________(7)

From equations (5), (6) and (7) we get, $\beta - \gamma = 0 \Rightarrow \beta = \gamma$

Put $\beta = \gamma$ in equation (4) then we get $\alpha = \gamma$.

Thus, if we take $\gamma = 1$.then we get $\beta = 1$ and $\alpha = 1$

Thus, we get $\alpha \neq 0$, $\beta \neq 0$ and $\gamma \neq 0$

Given Row vector are linearly dependent.

Now let remove R_3 Row and check the linearly independent or dependent for R_1= (4, 3,0, -2), R_2= (3, 4, -1, -3)

we have $\alpha, \beta \in R$ such that

α (4, 3,0, -2)+ β(3, 4, -1, -3) = (0, 0, 0)

4α +3 β = 0, 3α+4 β = 0, $-\beta$ = 0 and -2α -3 β = 0

we get, $\alpha = 0$ $\beta = 0$, $\gamma = 0$

Given Row vector R_1= (4, 3,0, -2), R_2= (3, 4, -1, -3)are linearly independent.

Rank of matrix A = r(A) = 2.

Solution: r(A)≤3

Let $C_1 = \begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}$, $C_2 = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ and $C_3 = \begin{bmatrix} 4 \\ 0 \\ 12 \end{bmatrix}$ be three column vectors Then

we have $\alpha, \beta, \gamma \in R$ such that

$$\alpha \begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix} + \beta \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} + \gamma \begin{bmatrix} 4 \\ 0 \\ 12 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

2α +3β +4γ = 0, 5α+2β +0γ = 0, and -4α + 5β +12γ = 0

2α +3β +4γ = 0,____________________(1)

5α+2β = 0, __________________(2)

-4α + 5β +12γ = 0 ___________________(3)

From equations (1) and (3)

$$\begin{aligned} 6\alpha + 9\beta + 12\gamma &= 0 \\ \underline{4\alpha - 5\beta - 12\gamma = 0} \\ 10\alpha + 4\beta \qquad &= 0 \quad \Rightarrow \; 5\alpha + 2\beta = 0 \;\_\_\_\_\_\_\_\_\_\_\_(4) \end{aligned}$$

From equations (2) and (4)

we get $5\alpha + 2\beta = 0 \Rightarrow \alpha = -\frac{2}{5}\beta$

Put $\alpha = -\frac{2}{5}\beta$ in equation (1) then we get $\gamma = -\frac{11}{20}\beta$.

Thus, if we take $\beta = 1$.then we get $\gamma = -\frac{11}{20}$ and $\alpha = -\frac{2}{5}$

Thus, we get $\alpha \neq 0$, $\beta \neq 0$ and $\gamma \neq 0$

Given column vectors are linearly dependent.

Now let remove C_3 column and check the linearly independent or dependent for $C_1 = \begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}$, $C_2 = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$

we have $\alpha, \beta \in R$ such that

$$\alpha \begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix} + \beta \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2\alpha + 3\beta = 0$, $5\alpha + 2\beta = 0$ and $-4\alpha + 5\beta = 0$

we get, $\alpha = 0$, $\beta = 0$

Given column vectors $C_1 = \begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}$, $C_2 = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ are linearly independent.

Rank of matrix A = r(A) = 2.