Sem-III
MAT 202: Linear Algebra-I
UNIT –2 Linearly dependence

Definition:- Trivial linear combination:

If $u_1, u_2, u_3, \ldots, u_n$ are n vectors of a vector space V, then the linear combination $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \ldots + \alpha_n u_n$ is called a trivial linear combination. If all the scalars $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are zero.

Definition:- Non-Trivial linear combination:

If $u_1, u_2, u_3, \ldots, u_n$ are n vectors of a vector space V, then the linear combination $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \ldots + \alpha_n u_n$ is called a non- trivial linear combination. If at least one of the scalars $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ is not zero. i.e. at least one of the $\alpha's$ is not zero.

e.g. (1) $0u_1+0u_2+0u_3+ \ldots +0u_n$ is a trivial linear combination.
(2) $0u_1+0u_2+0u_3+ \ldots +0u_{n-1} +1u_n$ and $1u_1+2u_2+3u_3+ \ldots +nu_n$ is a non trivial linear combination.

Note:- The trivial linear combination of any set of vectors is always the zero vector for $0u_1+0u_2+0u_3+ \ldots +0u_n = 0+0+0\ldots+0=0$

Example:- Give an example to show that a nontrivial linear combination of a set of vectors can give the zero vector.

Solution:- Example: Let (1, 0,0), (2, 0, 0) and (0, 0, 1) be three vectors in V_3. Then we have $\alpha, \beta, \gamma \in R$ such that $\alpha(1, 0,0)+\beta(2, 0, 0) +\gamma(0, 0, 1) = (0, 0, 0) = 0$

Thus we get $\alpha = 1$, $\beta = \frac{-1}{2}$ and $\gamma = 0$

i.e. $1(1, 0,0)+ \frac{-1}{2}(2, 0, 0) +0(0, 0, 1) = (0, 0, 0) = 0$

thus a nontrivial linear combination may give the zero vector.

Example :- Prove that (1, 0,0) is a linear combination of (2, 0, 0) and (0, 0, 1).

Solution:- Let $(1, 0,0) = \alpha(2, 0, 0) + \beta(0, 0, 1)$ $\alpha, \beta \in R$

$(1, 0,0) = (2\alpha, o, \beta)$

$\therefore 2\alpha = 1$ and $\beta = 0$

$\therefore \alpha = \frac{1}{2}$ and $\beta = 0$

$\therefore$ the linear combination of (2, 0, 0) and (0, 0, 1) is as

$(1, 0,0) = \frac{1}{2}(2, 0, 0) +0(0, 0, 1)$

Example:- Prove that the set of vectors { (1, 0, 0), (0, 1, 0), (0, 0, 1)} is trivial linear combination.

Solution:- Let $\alpha, \beta, \gamma \in R$ such that

α (1, 0,0)+ β (0, 1, 0) + γ (0, 0, 1) = (0, 0, 0)

$\therefore$ (α , β , γ) = (0, 0, 0)

$\therefore \alpha = \beta = \gamma = 0$

$\therefore$ given vectors are trivial linear combination.

Definition:- Linearly dependent(L.D.):-

A set { u_1, u_2, u_3, ….,u_n} of vectors is said to be linearly dependent(L.D.) if there exists a nontrivial linear combination of u_1, u_2, u_3, ….,u_n that equals the zero vector.

i.e. The linear combination $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \ldots + \alpha_n u_n = 0$ with at least one of scalars $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ is not zero. i.e. at least one of the α's is not zero.

Definition:- Linearly independent(L.I.):-

A set { u_1, u_2, u_3, ….,u_n} of vectors is said to be linearly independent(L.I.) if there exists a trivial linear combination of u_1, u_2, u_3, ….,u_n that equals the zero vector.

i.e. The linear combination $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \ldots + \alpha_n u_n = 0$ with all scalars α_1, $\alpha_2, \alpha_3, \ldots, \alpha_n$ zero. i.e. all of the α's is zero.

Example:- Prove that the vectors (1, 0,1), (1, 1, 0) and (-1, 0, -1) are L.D.

Solution:- Let $\alpha, \beta, \gamma \in R$ such that

α (1, 0,1)+ β (1, 1, 0) + γ (-1, 0, -1) = (0, 0, 0)

$\therefore$ ($\alpha + \beta - \gamma, \beta, \alpha + \beta$) = (0, 0, 0)

$\therefore \alpha + \beta - \gamma = 0, \quad \beta = 0, \; \alpha + \beta = 0$

Solving these equations then we get

$\therefore \beta = 0, \; \alpha = \gamma$

Thus any nonzero value for α ,say 1, then we get

1(1, 0,1)+0(1, 1, 0) +1(-1, 0, -1) = (0, 0, 0)

Hence this is a nontrivial linear combination of given vectors.

i.e. the vectors (1, 0,1), (1, 1, 0) and (-1, 0, -1) are L.D.

Example:- Prove that the vectors (1, 0,1), (1, 1, 0) and (1, 1, -1) are L.I.

Solution:- Let $\alpha, \beta, \gamma \in R$ such that

α (1, 0,1)+ β (1, 1, 0) + γ (1, 1, -1) = (0, 0, 0)

$\therefore$ ($\alpha + \beta + \gamma, \beta + \gamma, \alpha - \gamma$) = (0, 0, 0)

$\therefore \alpha + \beta + \gamma = 0$, $\beta + \gamma = 0$ and $\alpha - \gamma = 0$

$\therefore \alpha = \beta = \gamma = 0$

Hence this is a trivial linear combination of given vectors.

i.e. the vectors (1, 0,1), (1, 1, 0) and (-1, 0, -1) are L.I.

Example:- Check whether the following set of vectors is L.D. or L.I.

(1) {(1, 0,1), (1, 1, 0), (1, -1, 1), (1, 2, -3)}

(2) {e^x, e^{2x}} in $\mathcal{C}^{(\infty)}(-\infty, \infty)$.

(3) $\{x, |x|\}$ in $\mathcal{C}(-\infty,\infty)$.

Solution:- (1) {(1, 0,1), (1, 1, 0), (1, -1, 1), (1, 2, -3)}

Let $\alpha,\beta,\gamma,\delta \in R$ such that

α(1, 0,1)+β(1, 1, 0) +γ(1, 1, -1) +δ(1, 2, -3) = (0, 0, 0)

$\therefore(\alpha+\beta+\gamma+\delta, \beta-\gamma+2\delta, \alpha+\gamma-3\delta)$ = (0, 0, 0)

$\therefore \alpha+\beta+\gamma+\delta = 0, \beta-\gamma+2\delta=0, \alpha+\gamma-3\delta = 0$

Solving above equation we get

$\therefore \alpha=5\delta, \beta=-4\delta\ \gamma = -2\delta, \delta=\delta$

If we take $\delta=1$ then

$\alpha=5, \beta= -4\gamma = -2, \delta=1$

thus $\alpha=\beta=\gamma=\delta\neq 0$

Hence this is a nontrivial linear combination of given vectors.

i.e. the set of vectors {(1, 0,1), (1, 1, 0), (1, -1, 1), (1, 2, -3)}is L.D.

(2) $\{e^x, e^{2x}\}$ in $\mathcal{C}^{(\infty)}(-\infty,\infty)$.

Let $\alpha, \beta \in R$ such that $\alpha e^x+\beta\ e^{2x} = 0\ \ x\in(-\infty,\infty)$---------(1)

Differentiate the equation with respect to x, then we get

$\alpha e^x+2\beta\ e^{2x}=0$ ----------(2)

Solving equation (1) and (2) then we get

$\beta\ e^{2x}=0$. since $e^{2x}\neq 0$

$\therefore \beta= 0$.

And we get $\alpha = 0$

$\therefore \beta=\alpha= 0$.

Hence this is a trivial linear combination of given vectors.

i.e. the set of vectors $\{e^x, e^{2x}\}$ in $\mathcal{C}^{(\infty)}(-\infty,\infty)$ are L.I.

(3) $\{x, |x|\}$ in $\mathcal{C}(-\infty,\infty)$.

Let $\alpha, \beta \in R$ such that $\alpha x+\beta\ |x| = 0$

Since the function $|x|$ is not differentiable at zero.

$\therefore \alpha x+\beta\ |x| = 0$ holds for all $x\in$(-1, 1)

So choosing two different values of x say $x=\frac{1}{2}$ and $x=\frac{-1}{2}$ then we get

$\frac{\alpha}{2}+\frac{\beta}{2}=0$ and $\frac{-\alpha}{2}+\frac{\beta}{2}=0$

$\therefore \alpha=\beta= 0$

$\therefore$The set is LI over (-1, 1).

Definition:- The line through *v*:

Given a vector $v\neq 0$, the set of all scalar multiples of v is called the line through v.

Geometrically: In the case of V_1, V_2 and V_3. It is nothing but the straight line through the origin and v.

Definition: Collinear:-

Two vectors v_1 and v_2 are collinear if one of them lies in the line through the other.

Note:- 0 is collinear with any nonzero vector v.

Definition: Plane through v_1 and v_2 :-

Given Two vectors v_1 and v_2 which are not collinear, their span, namely $[v_1, v_2]$ is called the plane through v_1 and v_2.

Geometrically: In the case of V_2 and V_3. It is nothing but the plane passing through the origin and v_1 and v_2.

Definition: Coplanar:-

Three vectors v_1, v_2 and v_3 are coplanar if one of them lies in the plane through the other two. e.g. 0 is coplanar with every pair of non collinear vectors.

Example :- Prove that the vectors v and αv of a vector space V are collinear.

Solution:- Since αv is a scalar multiple of v .

$\therefore$ αv lies in the line through v.

The vectors v and are αv collinear.

Example :- Prove that the functions sin$x$ and cos$x$ in $\mathcal{F}$ (I) the collinear.

Solution:- Since sin$x$ (or cos$x$) is not a scalar multiple of cos$x$ (or *sinx*).

$\therefore$ neither of the two lies in the line through the other.

$\therefore$ The function sin$x$ and cos$x$ in $\mathcal{F}$ (I) are not collinear.

Note:- It spane , namely,[sin$x$,cos$x$]={ $\alpha \sin x + \beta \cos x / \alpha, \beta$ any scalar} is the plane through the vectors sin$x$ and cos$x$.

Example :- The function sin$x$, cos$x$,tan$x$ in $\mathcal{F}$ (I) are obviously not coplanar because none of them lies in the plane through the other two.

Example:- Prove that the functions $\cos^2 x$,$\sin^2 x$, $\cos 2x$ are coplanar.

Solution:- Since $\cos 2x = \cos^2 x - \sin^2 x$

$\therefore$ $\cos 2x$ lies in the plane through $\cos^2 x$ and $\sin^2 x$. also $\cos 2x$ is linear combination of $\cos^2 x$ and $\sin^2 x$.

$\therefore$ the functions $\cos^2 x$,$\sin^2 x$, $\cos 2x$ are coplanar.

Theorem:- Let V be any vector space. Then

(a) The set $\{v\}$ is LD iff $v = 0$.

(b) The set $\{v_1, v_2\}$ is LD iff v_1 and v_2 are coplanar.
 i.e. one of them is a scalar multiple of other.

(c) The set $\{v_1, v_2, v_3\}$ is LD iff v_1 ,v_2 and v_3 are coplanar.

i.e. one of them is a scalar multiple of other two.

Proof:- (a)The set $\{v\}$ is LD iff there exists a nonzero scalar α such that $\alpha v = 0$
Since $\alpha \neq 0 \Rightarrow v = 0$.

(b) suppose the set $\{v_1, v_2\}$ is L. D.
$\therefore$ there exist $\propto, \beta \in R$ with let $\propto \neq 0$ Such that
$\propto v_1 + \beta v_2 = 0$
$\therefore\ v_1 = -\frac{\beta}{\propto} v_2$
$\therefore\ v_1$ is scalar multiple of v_2.
$\therefore\ v_1$ is lies in the line through v_2.
$\therefore\ v_1, v_2$ are collinear.

Conversely,
let us suppose that v_1, v_2 are collinear.
$\therefore$ one of them say v_1 lies in the line through v_2.
$\therefore\ v_1$ is scalar multiple of v_2
$\therefore\ v_1 = \propto v_2$
i.e. $1.v_1 - \propto v_2 = 0$
since $1 \neq 0$
$\therefore\ v_1$ and v_2 are L. D.

(c) Let us suppose that $\{v_1, v_2, v_3\}$ is L. D.
$\therefore \propto, \beta, \gamma \in R$ with at least one of them say $\propto \neq 0$ Such that
$\propto v_1 + \beta v_2 + \gamma v_3 = 0$
$\therefore v_1 = \left(\frac{-\beta}{\alpha}\right) v_2 + \left(\frac{\gamma}{\alpha}\right) v_3$
i.e. $v_1 \in [v_1, v_3]$
$\therefore\ v_1$ lies in the plane through v_2 and v_3.
$\therefore\ v_1, v_2$ and v_3 are coplanar.

Conversely,
Let us suppose that v_1, v_2 and v_3 are coplanar
$\therefore$ one of them, say $v_1 \in [v_2, v_3]$
i.e. $v_1 = \propto_2 v_2 + \propto_3 v_3 \quad \forall \propto_2, \propto_3 \in R$.
$\therefore 1.v_1 - \alpha_2 v_2 - \alpha_3 v_3 = 0$
Since $1 \neq 0$
$\therefore\ v_1, v_2$ and v_3 are L. D.

Explain by illustration for above theorem.
Let us consider the three vectors (1,1,1), (1,-1,1) and (3,-1,3)
They are L. D.
Because
1(1,1,1)+2(1,-1,1)-1(3,-1,3)=0
$\therefore$ the plane through (1,1,1) and (3,-1,3) contains the point (1,-1,1).
As the plane through (1,1,1) and (3,-1,3) is
$[(1,1,1), (3,-1,3)] = \alpha(1,1,1) + \beta(3,-1,3)$
$\forall, \alpha, \beta \in R$

$$= \left\{\alpha + 3\beta, \alpha - \beta, \alpha + \frac{3\beta}{\alpha}, \beta \in R\right\}$$

Let (1,-1,1) ∈ [(1,1,1), (3,-1,3)]

$\therefore (1,-1,1) = \alpha(1,1,1) + \beta(3,-1,3)$

$\therefore \alpha + 3\beta = 1,\ \ \alpha - \beta = -1,\ \ \alpha + 3\beta = 1$

$\therefore \alpha = \frac{-1}{2}\ and\ \beta = \frac{1}{2}$

Note: In a vector space V any set of vectors containing the zero vector is L. D.

If $\{v_1, v_2 \ldots \ldots, v_n\}$ is a set and $v_1 = 0$

then $0v_1 + 0v_2 \ldots \ldots + 0v_{i-1} + 1v_i + 0v_{i+1} + \cdots + 0v_n$ is a nontrivial linear combination resulting in the zero vector.

Ex. In a vector space$V$, if v is a linear combination of $v_1, v_2 \ldots \ldots v_n$,

i.e. $v \in [v_1, v_2 \ldots \ldots v_n]$ then prove that $\{v_1, v_2 \ldots \ldots v_n\}$ is L. D.

Solution:

Since $v \in [v_1, v_2 \ldots \ldots v_n]$ is given

$\therefore v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_n v_n$

$\forall \alpha_i \in R\ i = 1,2,3 \ldots n$

i.e. $1.v - \alpha v_1 - \alpha v_2 \ldots \ldots - 2nv_n = 0$

Since $1 \neq 0$

$\therefore \{v_1, v_2 \ldots \ldots v_n\}$ is L. D.

Example:-In a vector space$V$, if the set $\{v_1, v_2 \ldots \ldots v_n\}$ L. I. and

$v\aleph \in [v_1, v_2 \ldots \ldots v_n]$ then prove $\{v_1, v_2 \ldots \ldots v_n\}$ is L. I.

Solution: Let us suppose that $v \in [v_1, v_2 \ldots \ldots v_n]$

$\therefore \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$

$\forall \alpha_i \in R,\quad i = 1,2,3, \ldots \ldots n$

$\therefore 1.v - \alpha_1 v_1 - \alpha_2 v_2 \ldots \ldots - \alpha_n v_n = 0$

Since $1 \neq 0$

$\therefore \{v_1, v_2 \ldots \ldots v_n\}$ is L. D.

Theorem:- (a) If a set is LI, then any subset of it is also LI and

(b) If a set is LD, then any superset of it is also LD.

Theorem:-In a vector space V. Suppose { $v_1, v_2, v_3, \ldots, v_n$} is an ordered set of vectors with $v_1 \neq 0$. The set is LD iff one of the vectors $v_2, v_3, \ldots, v_n$, say v_k, belongs to the span of $v_1, v_2, v_3, \ldots, v_{k-1}$

i.e. $v_k \in [v_1, v_2, v_3, \ldots, v_{k-1}]$ for some k = 1,2,3,…,n.

Proof:- Suppose $v_k \in [v_1, v_2, v_3, \ldots, v_{k-1}]$

i.e. v_k is a linear combination of $v_1, v_2, v_3, \ldots, v_{k-1}$.

$\therefore$ the set { $v_1, v_2, v_3, \ldots, v_{k-1}, v_k$ } is LD.

Since { $v_1, v_2, v_3, \ldots, v_n$}is a subset of the set { $v_1, v_2, v_3, \ldots, v_{k-1}, v_k$ }

$\therefore$ { $v_1, v_2, v_3, \ldots, v_{k-1}, v_k$ } is superset of { $v_1, v_2, v_3, \ldots, v_n$}

$\therefore$ { $v_1, v_2, v_3, \ldots, v_n$} is LD.

Conversely,

Let us suppose that { $v_1, v_2, v_3, \ldots, v_n$} is LD.

Now consider the set

$S_1 = \{v_1\}$

$S_2 = \{v_1, v_2\}$
$S_3 = \{v_1, v_2, v_3\}$
$\vdots$
$S_i = \{ v_1, v_2, v_3, \ldots, v_i\}$
$\vdots$
$S_n = \{ v_1, v_2, v_3, \ldots, v_n\}$
Here $S_1 = \{v_1\}$ is LI because $v_1 \neq 0$
But $S_n = \{ v_1, v_2, v_3, \ldots, v_n\}$ is LD is given.
So we go down the list and choose the first linearly dependent set.
Let S_k be first linearly dependent set.
i.e. S_k is linearly dependent set (LD) and S_{k-1} is linearly independent set(LI).
Here $2 \leq k \leq n$
Since S_k is LD
$\therefore \alpha_i \in R, i = 1,2,3,\ldots,k$ with at least one of $\alpha_i \neq 0$ such that
$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \ldots + \alpha_k v_k = 0$ ----------(1)
Let $\alpha_k \neq 0$
If $\alpha_k = 0$ then S_{k-1} would become a linearly dependent.
But S_{k-1} is LI.
$\therefore \alpha_k = 0$ is not possible.
$\therefore \alpha_k \neq 0$.
From equation (1) we get
$v_k = -\frac{\alpha_1}{\alpha_k} v_1 + -\frac{\alpha 2}{\alpha_k} v_2 + -\frac{\alpha_3}{\alpha_k} v_3 + \ldots + -\frac{\alpha_{k-1}}{\alpha_k} v_{k-1}$
v_k is linear combination of $\{ v_1, v_2, v_3, \ldots, v_{k-1}\}$
i.e. $v_k \in [v_1, v_2, v_3, \ldots, v_{k-1}]$

Corollary:- A finite subset $S = \{ v_1, v_2, v_3, \ldots, v_n\}$ of a vector space V containing a nonzero vector has a linearly independent subset A such that [A] = [S]

Proof:- Assume that $v_1 \neq 0$
If S is LI then there is nothing to prove as we have A =S. and
If S is not LI then we have a vector v_k such that $v_k \in [v_1, v_2, v_3, \ldots, v_{k-1}]$
Now discard v_k then the remaining set $S_1 = \{ v_1, v_2, v_{k-1}, v_{k+1}, \ldots, v_n\}$ has the same span as that of S.
If S_1 is LI then there is nothing to prove that and we have S_1= S.
And
If S_1 is not LI then repeat the foregoing process.
Then finally we get a linearly independent subset A such that [A] = [S]

Example:- Show that the ordered set {(1, 1,0), (0, 1, 1), (1, 0, -1), (1, 1, 1)}is L.D. and locate one of the vectors that belongs to the span of the previous ones. Also find the largest linearly independent subset whose span in $[S_4]$.

Solution:- Let us consider the sets
S_1 = { (1, 1,0)}
S_2 = {(1, 1,0)), (0, 1, 1)}
S_3 = {(1, 1,0), (0, 1, 1), (1, 0, -1)}

$S_4 = \{(1, 1,0), (0, 1, 1), (1, 0, -1), (1, 1, 1)\}$
Here $S_1 = \{ (1, 1,0)\}$ is LI $[\because (1, 1,0) \neq (0, 0,0)]$
Now $S_2 = \{(1, 1,0)), (0, 1, 1)\}$ is also LI because neither of the two vectors in S_2 is a scalar multiple of the other.
i.e. $(1, 1,0)) \neq \alpha\ (0, 1, 1)$ or $(0, 1,1)) \neq \alpha\ (1, 1, 0)$

{(or) for $\alpha, \beta \in R$ such that $\alpha\ (1, 1,0)) + \beta\ (0, 1, 1) = (0, 0, 0)$ thus we get $\alpha = \beta = 0$ there for $S_2 = \{(1, 1,0)), (0, 1, 1)\}$ is also LI }
Now $S_3 = \{(1, 1,0), (0, 1, 1), (1, 0, -1)\}$ is LD
because for $\alpha, \beta, \gamma \in R$ such that
$\alpha\ (1, 1,0)) + \beta\ (0, 1, 1) + \gamma\ (1, 0, -1) = (0, 0, 0)$
$\therefore \alpha + \gamma = 0,\ \alpha + \beta = 0,\ \beta - \gamma = 0$
From these equations we get $\alpha = -\gamma,\ \beta = \gamma,\ \gamma = \gamma$
Thus if we take $\gamma = 1$ then we get $\alpha = -1,\ \beta = 1,\ \gamma = 1$
$\therefore \alpha = \beta = \gamma \neq 0$
$\therefore S_3 = \{(1, 1,0), (0, 1, 1), (1, 0, -1)\}$ is LD
Hence $(1, 0, -1) \in [\{(1, 1,0), (0, 1, 1)]$
Now $S_4 = \{(1, 1,0), (0, 1, 1), (1, 0, -1), (1, 1, 1)\}$ is LD
Because $S_3 \subset S_4$ i.e. S_4 is super set of S_3.
Since $(1, 0, -1) \in [\{(1, 1,0), (0, 1, 1)]$
$\therefore (1, 0, -1) \in [\{(1, 1,0), (0, 1, 1), (1, 1, 1)]$
Now discard $(1, 0, -1)$ from the set S_4 then the span of the remaining set $A = \{(1, 1,0), (0, 1, 1), (1, 1, 1)\}$ is the same as $[S_4]$.
Let us check for the linear independent of A
Let $\alpha, \beta, \gamma \in R$ such that
$\alpha\ (1, 1,0)) + \beta\ (0, 1, 1) + \gamma\ (1, 1, 1) = (0, 0, 0)$
$\therefore \alpha + \gamma = 0,\ \alpha + \beta + \gamma = 0,\ \beta + \gamma = 0$
From these equations we get
$\therefore \alpha = \beta = \gamma = 0$
Set $A = \{(1, 1,0), (0, 1, 1), (1, 1, 1)\}$ is the largest linearly independent subset whose span in $[S_4]$. i.e. $[A] = [S_4]$

Note :- An infinite subset S of a vector space V is said to be linearly independent if every finite subset of S is LI. And an infinite subset S of a vector space V is said to be linearly dependent if it is not LI.

Example:- Prove that the subset $S = \{1, x, x^2, x^3, \ldots \}$ of $\mathcal{P}$ is LI
Solution:- Let $S_1 = \{1, x, x^2, x^3, \ldots x^n\}$
$a_1, a_2, a_3, \ldots, a_n \in R$ such that $a_1 1 + a_2 x + a_3 x^2 + \ldots + a_n x^n = o$
i.e. $a_1 1 + a_2 x + a_3 x^2 + \ldots + a_n x^n = 01 + 0 x + 0 x^2 + \ldots + 0 x^n$
Comparing the coefficient then we get
$a_1 = a_2 = a_3 = \ldots = a_n = 0$
$\therefore S_1$ is finite and LI.
Since S is infinite set but $S_1 \subset S$ and S_1 is LI
$\therefore$ S is LI.

Dimension and Basis

Definition: **Basis:-**

A subspace B of a vector space V is said to be a basis for V if

(a) B is linearly independent and

(b) [B]=V i.e. B generates V.

Example:- Prove that the set B = $\{i, j, k\}$ is basis for V_3 where i=(1,0,0), j=(0,1,0) and k=(0,0,1).

(a) Check set B for LI.

$\alpha(1,0,0)+\beta(0,1,0)+\gamma(0,0,1)=0$

$\therefore \alpha=0,\ \beta=0,\ \gamma=0$

$\therefore$ set B is LI.

(b) Let us check for set B as [B]=V

Let $(x,y,z)\in V_3$ and $(\alpha,\beta,\gamma)\in$ R such that

$\alpha(1,0,0)+\beta(0,1,0)+\gamma(0,0,1)=(x,y,z)$

$\therefore\ \alpha=x,\ \beta=y,\ \gamma=z$

$\therefore\ x(1,0,0)+y(0,1,0)+z(0,0,1)=(x,y,z)$

Which is the required linear combination of set B .

$\therefore$ [B] $=V_3$.

$\therefore$ Set B is Basis for V_3.

Example:- Prove that the set B = {(1,1,0), (1,0,1),(0,1,1)} is basis for V_3.

(a) Let $\alpha,\beta,\gamma\in R$ such that

$\alpha(1,1,0)+\beta(1,0,1)+\gamma(0,1,1)=(0,0,0)$

$\therefore \alpha+\beta=0,\ \beta+\gamma=0,\ \gamma+\alpha=0$

$\therefore \alpha=0,\ \beta=0,\ \gamma=0$

$\therefore$ set B is LI. -----------------(1)

(b) Now Let $(x,y,z)\in V_3$ and $(\alpha,\beta,\gamma)\in$ R such that

$\alpha(1,1,0)+\beta(1,0,1)+\gamma(0,1,1)=(x,y,z)$

$\therefore \alpha+\beta=x,\ \beta+\gamma=z,\ \gamma+\alpha=y$

$\therefore \alpha=\frac{x+y-z}{2},\ \beta=\frac{x-y+z}{2},\ \gamma=\frac{z-x+y}{2}$

$\frac{x+y-z}{2}(1,1,0)+\frac{x-y+z}{2}(1,0,1)+\frac{z-x+y}{2}(0,1,1)=(x,y,z)$

Which is linear combination of vector of set B.

i.e. B generates V_3.

$\therefore$ [B] = V_3. ------------------(2)

From (1) and (2)

the set B = {(1,1,0), (1,0,1),(0,1,1)} is basis for V_3.

Note: The above example shows that a basis for a vector V need not be unique.

Theorem:- In a vector space V if $\{v_1, v_2, v_3, \ldots, v_n\}$ generates V and if

$\{w_1, w_2, w_3, \ldots, w_n\}$ is LI then prove that $m \leq n$.

OR We cannot have more linearly independent vectors than the number of elements in a set of generators.

Proof:- Since $\{v_1, v_2, v_3, \ldots, v_n\}$ generates V.

i.e. $V = [v_1, v_2, v_3, \ldots, v_n]$

Let $w_1 \in V$

$\therefore w_1 \in [v_1, v_2, v_3, \ldots, v_n]$

$\therefore$ The new set $\{w_1, v_1, v_2, v_3, \ldots, v_n\}$ will be LD

($\because$ one of vector is a linear combination of other one)

Since the set is LD

$\therefore$ there exists a vector which is the linear combination of the preceding vectors.

Such vector must be from v_i's.

Let such a vector be v_j.

We discard this vector due to which the set becomes LD.

Remaining vectors will have same span.

i.e. $V = [w_1, v_1, v_2, v_3, \ldots v_{j-1}, v_{j+1,\ldots,} v_n]$

Again an element $w_2 \in V$ must be the linear combination of vectors in this span.

$\therefore w_2 \in [w_1, v_1, v_2, v_3, \ldots v_{j-1}, v_{j+1,\ldots,} v_n]$

Add the vector w_2 and consider the set $\{w_2, w_1, v_1, v_2, v_3, \ldots v_{j-1}, v_{j+1,\ldots,} v_n\}$

Again this set is Ld

Again remove the vector which belongs to the span of preceding vectors.

Such vector must be from v_i's.

Let such a vector be v_k.

We discard this vector due to which the set becomes LD.

Then we get following set

$\{w_2, w_1, v_1, v_2, v_3, \ldots, v_{k-1}, v_{k+1}, \ldots, v_{j-1}, v_{j+1,\ldots,} v_n\}$

This is LI and which generates vector span V.

We continue the process of adding the vectors from w_i set and removing the vector from v_i set.

At the time of addition of the vector from w_i set the set becomes LD and at the time of removal of vector from v_i set the set becomes LI.

We have to prove that $m \leq n$.

Let us suppose that $m > n$.

Then the vectors from v_i set will be exhausted first, After consuming all vectors in set when we add the vector from w_i set we will get LD set of vectors from w_i set only.

This is impossible.

Since $\{w_i\}_{i=m}$ I LI and hence its every subset is LI.

$\therefore$ our supposition is wrong.

Hence $m \leq n$ is true.

Corollary:- If V has a basis of n elements, then every set of p vectors with p>n is LD.

Proof:- Let $B = \{v_1, v_2, v_3, \ldots, v_n\}$ the basis for V and $A = \{u_1, u_2, u_3, \ldots, u_n\}$ be set of vectors in V with $p > n$

We have to prove that A is LD.

Assume that A is not LD.

i.e. A is LI

So A is the set of LI vectors in V and B is the set of LI generators in V.

$\therefore$ $p \leq n$.

Which contradicts the hypothesis p>n.

$\therefore$ our supposition is wrong.

$\therefore$ A must be LD.

Corollary:- If V has a basis of n elements, then every other basis for V also has n elements.

Proof:- Let $B_1 = \{ v_1, v_2, v_3, \ldots, v_n\}$ and $B_2 = \{ w_1, w_2, w_3, \ldots, w_n\}$ are two bases for V. then B_1 and B_2 are LI and $[B_1]$ =V and $[B_2]$ =V

We have to prove that m = n

Since $[B_1]$ =V and B_2 are LI

i.e. B_1 is the set of LI generators of V and B_2 is the set of LI vector in V.

$\therefore$ $m \leq n$.------------(1)

Since $[B_2]$ =V and B_1 are LI

i.e. B_2 is the set of LI generators of V and B_1 is the set of LI vector in V.

$\therefore$ $n \leq m$.------------(2)

From (1) and (2)

m= n

$\therefore$ Every basis of V contains vector.

i.e. Numbers of elements in a basis for vector space V is always constant.

Definition:- Dimension of a vector space:-

If a vector space V has a basis consisting of a finite number of elements in a basis is called the dimension of the space. The vector space V is called finite dimensional and is written as dimV.

Note:-

- If dimV = n then V is said to be n-dimensional.
- If V is not finite dimensional then it is called infinite dimensional.
- If V = V0 = {0} its dimension is taken to be zero.
- If a vector space V is n –dimensional then there exist n linearly independent vectors in V.
- If a vector space V is n –dimensional then every set of n+1 vectors in V is linearly dependent (LD) vectors in V.

Example:- Prove that V_2 is 2-dimensional space and V_3 is 3-dimensional space.

Solution:- Here $e_1 = (1,0)$ and $e_2= (0,1)$ in V_2 and $e_1 = (1,0,0)$,$e_2= (0,1,0)$ and $e_2= (0,0,1)$ in V_3.

Since $\{e_1, e_2\}$ is basis for V_2 and $\{e_1, e_2, e_3\}$ is basis for V_3.

$\therefore$ $\dim V_2 = 2$ and $\dim V_3 = 3$

Example:- Prove that $s = \{e_1, e_2, \ldots, e_n\}$ be a standard basis for V_n and find the dimension for V_n.(or) R^n.

Solution:- Let $\alpha_i \in R$, $1 \leq i \leq n$ such that

$\alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n = (0, 0, 0, \ldots, 0)$

$\therefore \alpha_1(1, 0, 0, \ldots,0) + \alpha_2\ (0, 1, 0, \quad \ldots,0) + \ldots + \alpha_n\ (0, 0, 0, \ldots,1) = (0, 0, 0, \ldots,0)$

$\therefore (\alpha_1, \alpha_2, \ldots \alpha_n) = (0, 0, 0, \ldots,0)$

$\therefore \alpha_1 = \alpha_2 = \ldots\ldots = \alpha_n = 0$

$\therefore$ The set s = $\{e_1, e_2, \ldots, e_n\}$ is LI. ______________(1)

Now let $(x_1, x_2, x_3, \ldots\ldots, x_n) \in V_n$ such that

$\alpha_1 e_1 + \alpha_2\ e_2 + \ldots + \alpha_n e_n = (x_1, x_2, x_3, \ldots\ldots, x_n)$

$\therefore \alpha_1(1, 0, 0, \ldots,0) + \alpha_2\ (0, 1, 0, \ldots,0) + \ldots + \alpha_n\ (0, 0, 0, \ldots,1) = (x_1, x_2, x_3, \ldots\ldots, x_n)$

$\therefore (\alpha_1, \alpha_2, \ldots \alpha_n) = (x_1, x_2, x_3, \ldots\ldots, x_n)$

$\therefore \alpha_1 = x_1, \alpha_2 = x_2, \ldots\ldots \alpha_n = x_n$

$\therefore \alpha_1 e_1 + \alpha_2\ e_2 + \ldots + \alpha_n e_n = (x_1, x_2, x_3, \ldots\ldots, x_n)$ is linear combination of $\{e_1, e_2, \ldots, e_n\}$

i.e.[S] = V ____________________(2)

From (1) and (2)

s = $\{e_1, e_2, \ldots, e_n\}$ be a standard basis for V_n.

$\therefore$ dimV_n= n.

Example :- Find the dimension of the space p_n .

Solution:- Every polynomial in p_n is a linear combination of the function $\{1, x, x^2, x^3, \ldots x^n\}$

Let $a_1, a_2, a_3, \ldots, a_n \in R$ such that $a_1 1 + a_2 x + a_3 x^2 + \ldots + a_n x^n = o$

i.e. $a_1 1 + a_2 x + a_3 x^2 + \ldots + a_n x^n = 01 + 0\ x + 0\ x^2 + \ldots + 0\ x^n$

Comparing the coefficient, then we get

$a_1 = a_2 = a_3 = \ldots = a_n = 0$

$\therefore \{1, x, x^2, x^3, \ldots x^n\}$ is LI.

$\therefore \{1, x, x^2, x^3, \ldots x^n\}$ is basis for p_n.

$\therefore$ dim p_n = n+1.

Theorem:-In an n-dimensional vector space V, any set of n linearly independent vector is a basis. OR. Prove that any set of n linearly independent vector is a basis In an n-dimensional vector space V.

Proof:-Suppose B = $\{v_1, v_2, v_3, \ldots, v_n\}$ is a set of n linearly independent vectors.

We want to prove that B is basis.

For this,

Since B is set of n linearly independent vectors is given.

$\therefore$ we have to only prove that [B]= V.

Let $v \in V$

Now consider the set $B_1 = \{v_1, v_2, v_3, \ldots, v_n, v\}$.

$\therefore B_1$ is a set containing n +1 vectors in dimensional vector space V.

$\therefore B_1$ is LD.

$\therefore$ there exists a vector in this set which belongs to the span of preceding vectors.

But such vectors cannot be v_i, i =1,2,3,….,n because $\{v_i,\}_{i=1}$ is LI.

So any vector $v \in V$ can be expressed as a linear combination of B.

$\therefore v \in [v_1, v_2, v_3, \ldots, v_n]$

$\therefore$[B]= V.

$\therefore$[B] is basis of V.

Example: Prove that the set {(1,1,1), (1,-1,1), (0,1,1)} is a basis for V_3.

Solution: Let $\alpha,\beta,\gamma \in R$ such that

$$\alpha(1,1,1)+\beta(1,-1,1)+\gamma(0,1,1)=0$$

$$\therefore \alpha+\beta,\alpha-\beta+\gamma,\alpha+\beta+\gamma)=(0,0,0)$$

$$\therefore \alpha+\beta=0,$$

$$\alpha-\beta=0$$

$$\alpha=\beta=\gamma=0$$

$\therefore$ {(1,1,1), (1,-1,1), (0,1,1)} is LI-----------------(1)

Now $\forall(x,y,z)\in V_3$ such that

$$\alpha(1,1,1)+\beta(1,-1,1)+\gamma(0,1,1)=(x,y,z)$$

$$\therefore \alpha+\beta=x,$$

$$\alpha-\beta+\gamma=y$$

$$\alpha+\beta+\gamma=z$$

$$\gamma=z-x$$

$$\therefore \alpha=\frac{x-2y-z}{2},\ \beta=\frac{z-x}{2},\ \gamma=z-x$$

$$\therefore \frac{x-2y-z}{2}(1,1,1)+\frac{z-x}{2}(1,-1,1)+z-x(0,1,1)=(x,y,z)$$

Which is linear combination of vector {(1,1,1), (1,-1,1), (0,1,1)}

$\therefore$ V_3=[(1,1,1), (1,-1,1), (0,1,1)] ----------------------(2)

From (1)and (2) given set is basis for V_3.

Theorem: In a vector space V. Let B = { $v_1, v_2, v_3, \ldots, v_n$} span V . Then the following two conditions are equivalent.

(a) { $v_1, v_2, v_3, \ldots, v_n$} is a linearly independent set.

(b) If $v\in$ V then the expression $\alpha_1 v_1+\alpha_2 v_2+\alpha_3 v_3+\ldots+\alpha_n v_n$ is unique.

Proof:- Let us assume that { $v_1, v_2, v_3, \ldots, v_n$} is a linearly independent set.

Now we want to prove that for $v\in$ V , then the expression $v=\alpha_1 v_1+\alpha_2 v_2+\alpha_3 v_3+\ldots+\alpha_n v_n$ is unique.

For this,

Let $v=\beta_1 v_1+\beta_2 v_2+\beta_3 v_3+\ldots+\beta_n v_n$ be another expression of $v\in$ V.

$\therefore \alpha_1 v_1+\alpha_2 v_2+\alpha_3 v_3+\ldots+\alpha_n v_n = \beta_1 v_1+\beta_2 v_2+\beta_3 v_3+\ldots+\beta_n v_n$

$\therefore (\alpha_1-\beta_1)v_1+(\alpha_2-\beta_2)v_2+(\alpha_3-\beta_3)v_3+\ldots+(\alpha_n-\beta_n)v_n = 0$

But { $v_1, v_2, v_3, \ldots, v_n$} is a linearly independent set.

$\therefore (\alpha_1-\beta_1)=(\alpha_2-\beta_2)=(\alpha_3-\beta_3)=\ldots=(\alpha_n-\beta_n)=0$

$\therefore \alpha_1=\beta_1,\ \alpha_2=\beta_2,\ \alpha_3=\beta_3,\ \ldots,\ \alpha_n=\beta_n$

Hence the expression v $=\alpha_1 v_1+\alpha_2 v_2+\alpha_3 v_3+\ldots+\alpha_n v_n$ is unique.

Conversely,

Let us assume that $v\in$ V then the expression $\alpha_1 v_1+\alpha_2 v_2+\alpha_3 v_3+\ldots+\alpha_n v_n$ is unique.

We want to prove that { $v_1, v_2, v_3, \ldots, v_n$} is a linearly independent set.

Let us suppose that { $v_1, v_2, v_3, \ldots, v_n$} is not a linearly independent set.

Then there exists scalars $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ (not all zero) with at least one non zero scalar satisfying

$\alpha_1 v_1+\alpha_2 v_2+\alpha_3 v_3+ \ldots+\alpha_n v_n =0$ -----------(1)

Also $0v_1+0v_2+0v_3+ \ldots+0v_n =0$ -----------(2)

$\therefore$(1) and (2) are two different linear combination for the vector $0\in$V.

This contradicts to our assumption.

$\therefore$ our supposition is wrong.

$\therefore\{ v_1, v_2, v_3, \ldots, v_n\}$ is a linearly independent set.

Note:- From above theorem we get that a set B is a basis for a vector space V iff [B] = V. and the expression for $v\in$V in terms of elements of B is unique.

Definition:- Coordinate vector:-

Let B=$\{ v_1, v_2, v_3, \ldots, v_n\}$ be an ordered basis for V. Then a vector $v\in$V can be written as v $=\alpha_1 v_1+\alpha_2 v_2+\alpha_3 v_3+ \ldots+\alpha_n v_n$. The vector $(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$ is called the coordinate vector of V relative to the ordered basis B.

- It is denoted by $[V]_B$.
- $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are called the coordinates of V relative to the ordered basis B.
- the coordinate vector of V relative to are simply called the the coordinate vector.

Example:- Find the coordinate vector of the vector (2, 3, 4, -1) of V_4 relative to the standard basis for V_4.

Solution:- Since$\{e_1, e_2, e_3, e_4\}$ is basis for V_4.

Where e_1 = (1,0,0, 0) ,e_2= (0,1,0, 0) , e_3= (0,0,1,0) and e_3= (0,0,0,1)

For a, b, c, d$\in$R such that

(2, 3, 4, -1) = ae_1+ b e_2+ce_3+de_4

= a(1,0,0, 0) + b (0,1,0, 0)+c(0,0,1,0)+d(0,0,0,1)

(2, 3, 4, -1) = (a, b, c, d)

$\therefore$ a = 2, b = 3, c= 4, d = -1

$\therefore$ (2, 3, 4, -1) = $2e_1$+ 3 e_2+$4e_3$+-$1e_4$

$\therefore$The coordinate vector of the vector (2, 3, 4, -1) of V_4 relative to the standard basis for V_4 is (2, 3, 4, -1)

$\therefore$(2, 3, 4, -1) = $[(2, 3, 4, -1)]_B$.

Example:- Find the coordinate vector of the vector (2, 3, 4, -1) of V_4 relative to the ordered basis B = {(1,1,0, 0), (0, 1,1,0), (0, 0,1,1), (1,0,0,0)} for V_4.

Solution:- Let $\alpha,\beta,\gamma,\delta\in R$ such that

(2, 3, 4, -1) = α(1,1,0, 0)+β (0, 1,1,0) +γ(0, 0,1,1) +δ(1,0,0,0)

$\therefore$(2, 3, 4, -1) = $(\alpha+\delta, \alpha+\beta, \beta+\gamma, \gamma)$

$\therefore \alpha+\delta = 2, \alpha+\beta = 3, \beta+\gamma =4, \gamma = -1$

Solving above equation we get

$\therefore \alpha = -2, \beta =5, \gamma = -1, \delta =4$

Hence the coordinates of (2, 3, 4, - 1) relative to the ordered basis B are -2,5,-1 and 4.

$\therefore$ (-2,5,-1, 4) = $[(2, 3, 4, -1)]_B$.

Theorem:- Let the set { $v_1, v_2, v_3, \ldots, v_k$} be a set of linearly independent subset of an n-dimensional vector space V. Then we can find vectors $v_{k+1}, v_{k+2}, \ldots, v_n$ in V such that the set {$v_1, v_2, v_3, \ldots V_k, v_{k+1,\ldots,} v_n$} is a basis for V.

OR

any set of linearly independent vectors of vector space can be extended to the basis or starting from any LI set of vectors in a vector space we can construct basis for it.

Proof:- Here V is n-dimensional vector space.

$\therefore$ it has n LI generators.

But number of LI vectors of V must be less than the number of LI generators of V.

$\therefore k \le n$.

If k = n then the set { $v_1, v_2, v_3, \ldots, v_n$} will be a set of n LI vectors in an n-dimensional vector space V.

If k< n then the set { $v_1, v_2, v_3, \ldots, v_k$} is not basis of V, because any basis of V should contain n elements.

i.e. $[v_1, v_2, v_3, \ldots, v_k] \ne V$

i.e. $[v_1, v_2, v_3, \ldots, v_k] \subset V$.

Hence there exists at least one vector v_{k+1} of V which does not belongs to the span of { $v_1, v_2, v_3, \ldots, v_k$}

i.e. $v_{k+1} \notin [v_1, v_2, v_3, \ldots, v_k]$

we enlarge our set by adding v_{k+1} to our set thus obtaining {$v_1, v_2, v_3, \ldots V_k, v_{k+1}$}.

Obviously this set of k+1 vectors is LI.

If k+1 = n then this set is basis for V.

If k+1 < n again we can find a vector of v_{k+2} outside the span of {$v_1, v_2, v_3, \ldots V_k, v_{k+1}$}.

We repeat this process till we get {$v_1, v_2, v_3, \ldots V_k, v_{k+1,\ldots,} v_n$} a basis of V.

Example:- Given two linearly independent vectors (1, 0, 1, 0) and (0, -1, 1, 0) of V_4. Find a basis for V_4 that includes these two vectors.

Solution:- Since [(1, 0, 1, 0) ,(0, -1, 1, 0)] = {α,-β,$\alpha+\beta$,0/α,β any scalars}.

Since the fourth coordinate is always zero for vectors in this span.

$\therefore$ (0, 0, 0, 1) is not in this span.

Thus we get an enlarged linearly independent set

{(1, 0, 1, 0) ,(0, -1, 1, 0), (0, 0, 0, 1) }

And whose span is

[(1, 0, 1, 0) ,(0, -1, 1, 0), (0, 0, 0, 1)]= {α,-β,$\alpha+\beta$,γ/α,β,γ any scalars}.

Now we have to identify one element outside this span.

Since the third coordinate in the elements of this span is always $\alpha+\beta$.

So we can find a vector for which this is not true.

Thus (1, -2, 0, 0) is not in the span of the earlier set.

So we have a set B = {(1, 0, 1, 0) ,(0, -1, 1, 0), (0, 0, 0, 1), (1, -2, 0, 0) } which is LI and it is a basis for V_4.

Let us check set B is basis .

Let $\alpha, \beta, \gamma, \delta \in R$ such that

α(1, 0,1,0)+ β(0,-1, 1, 0) + γ(0 0, 0,1) + δ(1, -2, 0,0) = (0, 0, 0,0)

$\therefore (\alpha + \delta, -\beta - 2\delta, \alpha + \beta, \gamma) = (0, 0, 0,0)$

$\therefore \alpha + \delta = 0,\ -\beta - 2\delta = 0,\ \alpha + \beta = 0,\ \gamma = 0$

Solving above equation we get

thus $\alpha = \beta = \gamma = \delta = 0$

i.e. the set of vectors {(1, 0,1), (1, 1, 0), (1, -1, 1), (1, 2, -3)}is L.I.

$\therefore$ set B = {(1, 0, 1, 0) ,(0, -1, 1, 0), (0, 0, 0, 1), (1, -2, 0, 0) } is LI and a basis for V_4.

Example:- Let {(1, 1, 1, 1) , (1, 2, 1, 2) }be a linearly independent subset of vector space V_4. Extend it to the basis for V_4.

Solution:- We have

[(1, 1, 1, 1) , (1, 2, 1, 2)] = {$\alpha + \beta, \alpha + 2\beta, \alpha + \beta, \alpha + 2\beta / \alpha, \beta$ any scalars}.

Since the first and third coordinates are equal for all vectors in the span.

$\therefore$ (0, 3, 2, 3) is not in the span.

Thus we have enlarged linearly independent set

{(1, 1, 1, 1) , (1, 2, 1, 2), (0, 3, 2, 3)}

And whose span is

[(1, 1, 1, 1) , (1, 2, 1, 2), (0, 3, 2, 3)] = {$\alpha + \beta, \alpha + 2\beta + 3\gamma, \alpha + \beta + 2\gamma, \alpha + 2\beta + 3\gamma / \alpha, \beta, \gamma$ any scalars}.

Obviously the vector (2, 6, 4, 5) is not in this span.

Hence the set {(1, 1, 1, 1) , (1, 2, 1, 2), (0, 3, 2, 3), (2, 6, 4, 5)} is LI and a basis for V_4.

Theorem:- Let U be a subspace of a finite-dimensional vector space V.Then prove that dim U $\leq$ dim V. Equality holds only when U = V.

Proof:- Let B = { $v_1, v_2, v_3, \ldots, v_n$} be basis for V. This generates V and has n elements.

i.e. there can be at most n linearly independent vectors in V and therefore in U.

$\therefore$ Any set of linearly independent vectors in U cannot more than n vectors.

$\therefore$ dim U $\leq$ dim V

Let dim U = dim V

Let B_1 = basis of U.

$\therefore$ is B_1 LI and span U.

$\therefore$ B_1 contains n LI vectors in U.

But any set of n LI vectors in V is basis for V ($\because$ dim V= n).

$\therefore$ B_1 is basis of V.

Theorem:- (Dimension theorem)If U and W are two subspaces of a finite dimensional vector space V, then prove that dim(U + W) = dim U + dim W – dim U $\cap$W.

Proof:-Let dim U = m, dim W = p ,dim U $\cap$ W = r and dim V= n.

Since U, W and U$\cap$W are subspaces of a vector space V.

Since dimension of a subspace of a vector space cannot exceed that of a vector space.

$\therefore m \le n, \quad p \le n, \quad r \le n$

Let $\{ v_1, v_2, v_3, \ldots, v_r\}$ be basis for $U \cap W$.

Since vectors in basis is LI.

$\therefore \{ v_1, v_2, v_3, \ldots, v_r\}$ is LI in $U \cap W$

Hence $\{ v_1, v_2, v_3, \ldots, v_r\}$ is LI in U and W both.

In particular $\{ v_1, v_2, v_3, \ldots, v_r\}$ are LI vectors in U.

So that can set can be extended to the basis of U.

Thus we can find the vector $u_{r+1}, u_{r+2}, \ldots, u_m$ in U such that

$\{ v_1, v_2, v_3, \ldots, v_r, u_{r+1}, u_{r+2}, \ldots, u_m \}$ is a basis for U

Similarly, the set $\{ v_1, v_2, v_3, \ldots, v_r\}$ are LI vectors in W.

So that can set can be extended to the basis of W.

Thus we can find the vector $w_{r+1}, w_{r+2}, \ldots, w_p$ in W such that

$\{ v_1, v_2, v_3, \ldots, v_r, w_{r+1}, w_{r+2}, \ldots, w_p \}$ is a basis for W.

Now let us consider the set

$A = \{ v_1, v_2, v_3, \ldots, v_r, u_{r+1}, u_{r+2}, \ldots, u_m, w_{r+1}, w_{r+2}, \ldots, w_p \}$

We shall show that A is basis for U + W.

A will be basis for U + W if

(i) A is LI. (ii) [A] = U + W

Let x be any vector in U + W.

Then $x = u + w$ where $u \in U$ and $w \in W$.

But $u \in U$ is a linear combination of vectors $\{ v_1, v_2, v_3, \ldots, v_r, u_{r+1}, u_{r+2}, \ldots, u_m\}$

Similarly, $w \in W$ is a linear combination of vectors $\{v_1, v_2, v_3, \ldots, v_r, w_{r+1}, w_{r+2}, \ldots, w_p\}$

$\therefore$ $x = u + w$ becomes linear combination of A.

Hence [A] = U + W

To prove linear independence of A

Let $\alpha_i, \beta_j, \gamma_k$ for $1 \le i \le r$, $r+1 \le j \le m$, $1+r \le k \le p$ are scalars such that

$$\sum_{i=1}^{r} \alpha_i v_i + \sum_{i=r+1}^{m} \beta_i u_i + \sum_{i=r+1}^{p} \gamma_i w_i = 0$$

i.e $\sum_{i=1}^{r} \alpha_i v_i + \sum_{i=r+1}^{m} \beta_i u_i = -\sum_{i=r+1}^{p} \gamma_i w_i = v$ say -------------(1)

Since vectors $\{ v_1, v_2, v_3, \ldots, v_r, u_{r+1}, u_{r+2}, \ldots, u_m\}$ in U and

vector $w_{r+1}, w_{r+2}, \ldots, w_p$ in W.

vector v belongs both U and W.

i.e. $v \in U \cap W$.

$\therefore v$ can be expressed as a linear combination of basis of basis of $U \cap W$.

$\therefore v = \sum_{i=1}^{r} \delta_i v_i$ ------------------------(2)

From (1) and (2) we get

$\sum_{i=r+1}^{p} \gamma_i w_i + \sum_{i=1}^{r} \delta_i v_i = 0$ ------------------(3)

Since $\{v_1, v_2, v_3, .., v_r, w_{r+1}, w_{r+2}, \ldots ,w_p\}$ is LI.

Each γ_i and δ_i are zero.

In particular each γ_i i= r+1, ….,p is zero.

Using this with equation (1) we get

$$\sum_{i=1}^{r}\alpha_i v_i + \sum_{i=r+1}^{m}\beta_i u_i = 0$$

But $\{ v_1, v_2, v_3, \ldots, v_r, u_{r+1}, u_{r+2}, \ldots ,u_m\}$ be basis in U therefore it is LI.

Each α_i =0 for i= 1,2,…,r and β_i=0 for i= r+1,…., m

Thus we get the value of each scalars is zero.

$\therefore$ the set A is LI.

$\therefore$ set A is basis for U + W.

Hence dim(U + W) = m + p - r

= dim U + dim W – dim U $\cap$ W.

i.e. dim(U + W) = dim U + dim W – dim U $\cap$ W.