Unit -4Matrices -2

- **Definition: The eigen values and eigen vectors of a square matrix:**

Let A =$[a_{ij}]_n$ be a square matrix of order n and λ a scalar (real or complex). If there exists a matrix X, satisfying A X = λ X, then λ is called an eigen value of A and X is called an eigen vector of A corresponding to λ. (Here X is an n × 1 Matrix)

i.e. Let A = $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ and X = $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ then

A X = λ X ⇒A X -λ X = 0

⇒ (A – λ I_n) X = 0__________________(1)

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn}-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

In equation (1), |A – λ I_n| = 0 is called the characteristic equation of the matrix A.

Note: - If $[a_{ij}]_n$ be a square matrix of order n then the characteristic equation of the matrix A of degree n in λ. Its n roots give its eigen values of A. Putting these values of λ in equation (1) i.e.⇒ (A – λ I_n) X = 0 we get the corresponding eigen vectors of A.

A square matrix of order n can at most have n eigen values.

Example: - If $[a_{ij}]_n$ be a square matrix of order n and corresponding eigen value of a matrix exists then prove that it is unique.

Prove: - Let $[a_{ij}]_2$be a square matrix of order 2.

Let us suppose that λ_1 and $\lambda_2$two eigen values of corresponding to the eigen vector X.

$\therefore$A X -λ_1 X = 0 and A X -λ_2 X = 0

$\therefore$A X =λ_1 X and A X =λ_2 X

Thus, we get, λ_1 X =λ_2 X $\Rightarrow\lambda_1 \;\; =\lambda_2$

i.e. this prove that eigen value of a matrix is unique.

Example: -Find the Eigen values for the matrixA=$\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$.

Solution: -We know that for any given matrix|A-λI|=0

[A-λI]=$\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}-\lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}-\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ =0

[A-λI]=$\begin{bmatrix} 1-\lambda & -3 \\ 2 & 1+\lambda \end{bmatrix}$=(1-λ)(1+λ)+6=0

$\Rightarrow$1-λ^2+6= 0 (Which is the characteristic equation of the given matrix.)

$\Rightarrow$-λ^2+7= 0 $\Rightarrow\lambda^2$=7

$\therefore\lambda=\pm\sqrt{7}$

$\therefore\lambda=-\sqrt{7}$ or $\lambda=\sqrt{7}$ which are eigen values of given matrix A.

Example: -Find the Eigen values and eigen vectors for the matrixA=$\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$.

Solution: -We know that for any given matrix

[A-λI]X=0 and |A-λI|=0

[A-λI]=$\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}-\lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}-\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ =0

[A-λI]=$\begin{bmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{bmatrix}$=(1-λ)(2-λ)-12=0

$\Rightarrow\lambda^2$-3λ - 10= 0 (Which is the characteristic equation of the given matrix.)

$\Rightarrow$(λ-5)(λ+2) =0

∴λ=5 or λ= -2 which are eigen values of given matrix A.

Now we will find the eigen vectors corresponding to the eigen values λ=5 or λ= -2

For λ=5,

The equation for eigen vector as

[A-5I]X=0

$$\Rightarrow \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -4 & 4 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, we get,

-4x_1 +4x_2 =0 and 3x_1 -3x_2 =0

From these two equations we get, $x_1 = x_2$

Hence, the eigen vector corresponding to λ=5 is $\begin{bmatrix} a \\ a \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix} \ \forall a \in R - \{0\}$

For λ=-2,

The equation for eigen vector as

[A-5I]X=0

$$\Rightarrow \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, we get,

3x_1 +4x_2 =0 and 3x_1 +4x_2 =0

From these two equations we get, $x_1 = -\frac{4}{3}x_2$

Hence, the eigen vector corresponding to λ=-2 is $\begin{bmatrix} -\frac{4}{3}a \\ a \end{bmatrix} = a\begin{bmatrix} -\frac{4}{3} \\ 1 \end{bmatrix}$ $\forall a \in R - \{0\}$

Example: -Find the Eigen values and eigen vectors for the matrixA=$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution: -We know that for any given matrix

[A-λI]X=0 and |A-λI|=0

$$[A-\lambda I]=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}-\lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}-\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}=0$$

$$[A-\lambda I]=\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}= 0$$

∴ $(1-\lambda)^3=0$ (Which is the characteristic equation of the given matrix.)

∴λ=1which is eigen values of given matrix A.

Now we will find the eigen vectors corresponding to the eigen values λ=1

For λ=1,

The equation for eigen vector as

[A-Iλ]X=0

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, we get,

$0x_1 + 0x_2 + 0x_3 = 0$

From these two equations we get, $x_1 = a, x_2 =$ b and $x_3 =$ c, where a, b and c are any non zero real numbers.

Hence, the eigen vector corresponding to λ=1 is

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \forall a, b, c \in R - \{0\}$$

Note: - the eigen value of the identity matrix I_n is one (1).

Example: **-**Find the Eigen values and eigen vectors for the matrix$A=\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Solution: **-**We know that for any given matrix

[A-λI]X=0 and |A-λI|=0

$$[A-\lambda I]=\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = 0$$

$$[A-\lambda I]=\begin{bmatrix} -1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} = 0$$

∴ (-1-λ)(2-λ)(3-λ) =0 (Which is the characteristic equation of the given matrix.)

∴λ=-1,2,3which is eigen values of given matrix A.

Now we will find the eigen vectors corresponding to the eigen values λ=-1,2,3

For λ=-1,

The equation for eigen vector as

[A-Iλ]X=0

$$\Rightarrow \begin{bmatrix} -1+1 & 0 & 0 \\ 0 & 2+1 & 0 \\ 0 & 0 & 3+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, we get,

$0x_1 + 0x_2 + 0x_3 = 0$,$0x_1 + 3x_2 + 0x_3 = 0$ and $0x_1 + 0x_2 + 4x_3 = 0$

From these two equations we get, $x_1 = a, x_2 = 0$ and $x_3 = 0$, where a is any non-zero real numbers.

Hence, the eigen vector corresponding to λ=-1 is

$$\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \ \forall a, \in R - \{0\}$$

For λ=2,

The equation for eigen vector as

[A-Iλ]X=0

$$\Rightarrow \begin{bmatrix} -1-2 & 0 & 0 \\ 0 & 2-2 & 0 \\ 0 & 0 & 3-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, we get,

$-3x_1 + 0x_2 + 0x_3 = 0$,$0x_1 + 0x_2 + 0x_3 = 0$ and $0x_1 + 0x_2 + 1x_3 = 0$

From these two equations we get, $x_1 = 0, x_2 =$ a and $x_3 = 0$, where a is any non-zero real numbers.

Hence, the eigen vector corresponding to λ=2 is

$\begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix}$ $=a \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\forall a, \in R - \{0\}$

For λ=3,

The equation for eigen vector as

[A-Iλ]X=0

$$\Rightarrow \begin{bmatrix} -1-3 & 0 & 0 \\ 0 & 2-3 & 0 \\ 0 & 0 & 3-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, we get,

$-4x_1$ +0x_2 +0x_3 =0 ,0x_1 -1x_2 +0x_3 =0 and 0x_1 +0x_2 +0x_3 =0

From these two equations we get, $x_1 = 0, x_2$= 0 and x_3= a, where a is any non-zero real numbers.

Hence, the eigen vector corresponding to λ=3 is

$$\begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} = a \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \ \forall a, \in R - \{0\}$$

Note: -(1) the eigen value of a diagonal matrix areits diagonal elements.

(2) The eigen value of the matrix A =$[a_{ij}]_n$ is the same as the eigen values of its transpose A^T.

Example: -Find the Eigen values and eigen vectors for the matrixA=$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$.

Solution: -We know that for any given matrix

[A-λI]X=0 and |A-λI|=0

$$[A-\lambda I]=\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}-\lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}-\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}=0$$

$$[A-\lambda I]=\begin{vmatrix}2-\lambda&1&1\\0&1-\lambda&0\\1&1&2-\lambda\end{vmatrix}=0$$

$\therefore (2-\lambda)(1-\lambda)(2-\lambda)+1(0-(1-\lambda))=0$ (Which is the characteristic equation of the given matrix.)

$\therefore (2-\lambda)(1-\lambda)(2-\lambda)-(1-\lambda)=0$

$\therefore (1-\lambda)[(2-\lambda)^2-1]=0$

$\therefore (1-\lambda)[(2-\lambda-1)(2-\lambda+1)]=0$

$\therefore (1-\lambda)[(1-\lambda)(3-\lambda)]=0$

$\therefore \lambda=1,3$ which is eigen values of given matrix A.

Now we will find the eigen vectors corresponding to the eigen values $\lambda=1,3$

For $\lambda=1$,

The equation for eigen vector as

$[A-I\lambda]X=0$

$$\Rightarrow \begin{bmatrix}2-1&1&1\\0&1-1&0\\1&1&2-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}1&1&1\\0&0&0\\1&1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Thus, we get,

$x_1+x_2+x_3=0$, $0x_1+0x_2+0x_3=0$ and $x_1+x_2+x_3=0$

From these equations we get, x_1 =-(x_2 +$x_{3)}$if we take$x_2 = a$ and x_3=b, then we get $x_1 = -(a + b)$where a and bare any non-zero real numbers.

Hence, the eigen vector corresponding to λ=1 is

$$\begin{bmatrix} -(a+b) \\ a \\ b \end{bmatrix} = a\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \ \forall a, b \in R - \{0\}$$

For λ=3,

The equation for eigen vector as

[A-Iλ]X=0

$$\Rightarrow \begin{bmatrix} 2-3 & 1 & 1 \\ 0 & 1-3 & 0 \\ 1 & 1 & 2-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 1 \\ 0 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, we get,

$-x_1$ +x_2 +x_3 =0 ,0x_1-2x_2 +0x_3 =0 and x_1 +x_2-x_3 =0

From these equations we get, $x_1 = x_3, x_2$= 0 if we take x_3= a then we get,$x_1 = a$ where a is any non-zero real numbers.

Hence, the eigen vector corresponding to λ=3 is

$$\begin{bmatrix} a \\ 0 \\ a \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \ \forall a, \in R - \{0\}$$

Example: -Find the Eigen values and eigen vectors for the matrixA=$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$.

Solution: -We know that for any given matrix

[A-λI]X=0 and |A-λI|=0

$$[A\text{-}\lambda I]=\begin{bmatrix}2 & 1 & 0\\ 0 & 1 & -1\\ 0 & 2 & 4\end{bmatrix}-\lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}2 & 1 & 0\\ 0 & 1 & -1\\ 0 & 2 & 4\end{bmatrix}-\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix}=0$$

$$[A\text{-}\lambda I]=\begin{vmatrix}2-\lambda & 1 & 0\\ 0 & 1-\lambda & -1\\ 0 & 2 & 4-\lambda\end{vmatrix}=0$$

$\therefore (2-\lambda)[(\lambda)^2-5\lambda+6]=0$ (Which is the characteristic equation of the given matrix.)

$\therefore (2-\lambda)(\lambda-3)(\lambda-2)=0$

$\therefore \lambda=2,3$ which is eigen values of given matrix A.

Now we will find the eigen vectors corresponding to the eigen values $\lambda=2,3$

For $\lambda=2$,

The equation for eigen vector as

$[A\text{-}I\lambda]X=0$

$$\Rightarrow \begin{vmatrix}2-2 & 1 & 0\\ 0 & 1-2 & -1\\ 0 & 2 & 4-2\end{vmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}0 & 1 & 0\\ 0 & -1 & -1\\ 0 & 2 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Thus, we get,

$0x_1+x_2+0x_3=0$, $0x_1-x_2-x_3=0$ and $0x_1+2x_2+2x_3=0$

From these equations we get, $x_2=0$, $x_3=0$ if we take $x_1=a$ where a is any non-zero real numbers.

Hence, the eigen vector corresponding to $\lambda=2$ is

$$\begin{bmatrix}a\\ 0\\ 0\end{bmatrix}=a\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}\ \forall a\in R-\{0\}$$

For λ=3,

The equation for eigen vector as

[A-Iλ]X=0

$\Rightarrow \begin{vmatrix} 2-3 & 1 & 0 \\ 0 & 1-3 & -1 \\ 0 & 2 & 4-3 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Thus, we get,

$-x_1$ +x_2 +0x_3 =0 ,0x_1 -2x_2-x_3 =0 and 0x_1 +2x_2+x_3 =0

From these equations we get, $x_1 = x_2, x_3$= -2x_2 if we take x_2= a then we get,$x_1 = a$ and x_3= -2a where a is any non-zero real numbers.

Hence, the eigen vector corresponding to λ=3 is

$\begin{bmatrix} a \\ a \\ -2a \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \; \forall a, \in R - \{0\}$

Example: -Find the Eigen values and eigen vectors for the matrixA=$\begin{bmatrix} 0 & -2 & -2 \\ -2 & -3 & -2 \\ 3 & -6 & 5 \end{bmatrix}$.

Solution: -We know that for any given matrix

[A-λI]X=0 and |A-λI|=0

[A-λI]=$\begin{bmatrix} 0 & -2 & -2 \\ -2 & -3 & -2 \\ 3 & -6 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -2 \\ -2 & -3 & -2 \\ 3 & -6 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$ =0

[A-λI]=$\begin{bmatrix} 0-\lambda & -2 & -2 \\ -2 & -3-\lambda & -2 \\ 3 & -6 & 5-\lambda \end{bmatrix} = 0$

$\therefore -\lambda[(-3-\lambda)(5-\lambda)$-12]+2[-2$(5-\lambda)$+6]-2[12-3$(-3-\lambda)$] =0

$\therefore -\lambda[\lambda^2-2\lambda-27]$+2[$2\lambda-4$]-2[$(21+3\lambda)$] =0

$\therefore [-\lambda^3+2\lambda^2+27\lambda]$+[$4\lambda-8$]+[$(-42-6\lambda)$] =0

$$\therefore -\lambda^3+2\lambda^2+25\lambda-50=0$$

$\therefore \lambda^3-2\lambda^2-25\lambda+50=0$ (This is the characteristic equation of the given matrix.)

$\therefore (\lambda-2)[(\lambda)^2-25]$ =0

$\therefore (\lambda-2)(\lambda-5)(\lambda+5)$ =0

∴λ=2,5,-5which are the eigen values of given matrix A.

Another method to find eigen value

$$\boldsymbol{\lambda^3-D_1\lambda^2+D_2\lambda-|A|=0}$$

Where, D_1 = 0 +(-3) + 5 = 2 The diagonal element of matrix.

Where, D_2 = $\begin{vmatrix}-3 & -2\\-6 & 5\end{vmatrix}$+ $\begin{vmatrix}0 & -2\\3 & 5\end{vmatrix}$+ $\begin{vmatrix}0 & -2\\-2 & -3\end{vmatrix}$=(-15-12)+6-4 = -25

And $|A|=\begin{vmatrix}0 & -2 & -2\\-2 & -3 & -2\\3 & -6 & 5\end{vmatrix}$= -50

$$\boldsymbol{\lambda^3-2\lambda^2-25\lambda+50=0}$$

Now we will find the eigen vectors corresponding to the eigen values λ=2,5,-5

For λ=2,

The equation for eigen vector as

[A-Iλ]X=0

$\Rightarrow \begin{bmatrix}0-2 & -2 & -2\\-2 & -3-2 & -2\\3 & -6 & 5-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$\Rightarrow$

Thus, we get,

$-\lambda x_1$ -2x_2 -2x_3 =0 ,-2$x_1 + (-3-\lambda)x_2$ -2x_3 =0 and 3x_1-6x_2 +$(5-\lambda)x_3$ =0

For λ=2,

$-2x_1$ -2x_2 -2x_3 =0 ,-2$x_1 + (-3-2)x_2$ -2x_3 =0 and 3x_1 -6x_2 +$(5-2)x_3$ =0

x_1 +x_2 +x_3 =0 ,-2$x_1 - 5x_2$ -2x_3 =0 and x_1 -2x_2 +x_3 =0

we get,x_2 =0 $x_3 + x_1 = 0$

From these equations we get,x_2 =0,$x_3 = -x_1$ if we take$x_1 = a$ then we get $x_3 = -a$where a is any non-zero real numbers.

Hence, the eigen vector corresponding to λ=2 is

$$\begin{bmatrix} a \\ 0 \\ -a \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \ \forall a \in R - \{0\}$$

For λ=5,

$-5x_1$ -2x_2 -2x_3 =0 ,-2$x_1 + (-3-5)x_2$ -2x_3 =0 and 3x_1 -6x_2 +$(5-5)x_3$ =0

$-5x_1$ -2x_2 -2x_3 =0 ,-2$x_1 - 8x_2$ -2x_3 =0 and 3x_1 -6x_2 +0x_3 =0

$-5x_1$ -2x_2 -2x_3 =0 ______(1) ,$x_1 + 4x_2$ +x_3 =0_____(2) and x_1 -2x_2 =0 _______(3)

From (1) and (2) we get $x_1 = 2x_2$ and put the value $x_1 = 2x_2$ in equation (1) or (2) then we get $x_3 = -6x_2$

if we take$x_2 = a$ then we get$x_1 = 2a$ and $x_3 = -6a$ where a is any non-zero real numbers.

Hence, the eigen vector corresponding to λ=5 is

$$\begin{bmatrix} 2a \\ a \\ -6a \end{bmatrix} = a \begin{bmatrix} 2 \\ 1 \\ -6 \end{bmatrix} \ \forall a \in R - \{0\}$$

For λ=-5,

$5x_1$ -$2x_2$ -$2x_3$ =0 ,-$2x_1 + (-3+5)x_2$ -$2x_3$ =0 and $3x_1$ -$6x_2$ +$(5+5)x_3$ =0

$5x_1$ -$2x_2$ -$2x_3$ =0 ,-$2x_1 + 2x_2$ -$2x_3$ =0 and $3x_1$ -$6x_2$ +$10x_3$ =0

$5x_1$ -$2x_2$ -$2x_3$ =0 _____(1) ,$x_1 - x_2 + x_3$ =0____(2) and $3x_1$ -$6x_2$ +$10x_3$ =0

______(3)

Remove $x_1$from (1) , (2) and (3) then we get $3x_2 = 7x_3$ and put the value in equation (1) or (2) or (3) then we get $3x_1 = 4x_3$

if we take$x_3 = a$ then we get$x_1 = \frac{4}{3}a$ and $x_2 = \frac{7}{3}a$ where a is any non-zero real numbers.

Hence, the eigen vector corresponding to λ=5 is

$$\begin{bmatrix} \frac{4}{3}a \\ \frac{7}{3}a \\ a \end{bmatrix} = a\begin{bmatrix} \frac{4}{3} \\ \frac{7}{3} \\ 1 \end{bmatrix} \ \forall a \in R - \{0\}$$

Example: - If $\lambda_1, \lambda_{2,}\lambda_3, \dots, \lambda_n$ are the eigen values of matrix A =$[a_{ij}]_n$, then prove that ${\lambda_1}^3, {\lambda_{2,}}^3{\lambda_3}^3, \dots, {\lambda_n}^3$ are the eigen values of A^3.

Solution: - Here A^n= A.A.A … A (n times).

Hence A^3= A^2.A=A.A.A

Let λ be an eigen value of A. therefore there exist a non-zero column matrix X such that

AX = λX ___________________(1)

⇒A^2AX = $A^2(\lambda X)$

⇒A^3X = $\lambda A^2 X$__________________(2)

But $A^2 X$ = A(AX) =A(λX)=λ(AX)=$\lambda(\lambda X)$=$\lambda^2 X$(∵ from equation (1))

Thus, we get $A^2X = \lambda^2 X$______________(3)

Put the value of equation (3) in equation (2) then, we get $A^3X = \lambda^3 X$

$\therefore$ λ^3 is the eigen value of A^3.

Thus, if $\lambda_1, \lambda_{2,}\lambda_3, \ldots, \lambda_n$ are the eigen values of matrix $A = [a_{ij}]_n$, then $\lambda_1{}^3, \lambda_{2,}{}^3\lambda_3{}^3, \ldots, \lambda_n{}^3$ are the eigen values of A^3.

Example: - If $\lambda$is the eigen values of matrix $A = [a_{ij}]_n$, then prove that

(i) $\frac{1}{\lambda}$is the eigen values of A^{-1}. (ii) $\frac{|A|}{\lambda}$ is the eigen values of *adj* A.

Solution: -

Let λ be an eigen value of A. therefore, there exist a non-zero column matrix X such that $AX = \lambda X$

$\therefore$ $X = A^{-1}\lambda X = \lambda A^{-1}X$

$\therefore$ $\frac{1}{\lambda} X = A^{-1}X$

This prove that $\frac{1}{\lambda}$ is the eigen values of A^{-1}.

(ii) Let λ be an eigen value of A. therefore, there exist a non-zero column matrix

X such that $AX = \lambda X$

$\Rightarrow adj\ A\ (AX) = adj\ A\ (\lambda X)$

$\Rightarrow (adj\ A\ A)(X) = \lambda\ (adj\ A)X$

$$\left(because\ A^{-1} = \frac{adj\ \mathrm{A}}{|A|} = adj\ \mathrm{A} = A^{-1}|A| \Longrightarrow adj\ \mathrm{A}\ \mathrm{A} = |A|\ \right)$$

$\Rightarrow |A|\ (X) = \lambda\ (adj\ A)\ X$

$\Rightarrow \frac{|A|}{\lambda}(X) = (adj\ A)\ X$

This prove that $\frac{|A|}{\lambda}$ is the eigen values of *adj* A.

Theorem: (Cayley- Hamilton Theorem) (without proof)

Every square matrix A = $[a_{ij}]_n$satisfies its own characteristic equation. i.e.
If$|A - \lambda I_n| = (-1)^2\lambda^n + k_1\lambda^{n-1} + k_2\lambda^{n-2} + \cdots + k_n = 0$ is the characteristic equation of A, then $(-1)^2A^n + k_1A^{n-1} + k_2A^{n-2} + \cdots + k_n = 0$

Example: Verify Calay-Hamilton theorem for the matrix A = $\begin{bmatrix} 2 & -4 & -1 \\ 0 & 3 & 4 \\ 1 & 6 & 2 \end{bmatrix}$. Also using this theorem find A^{-1}.

Solution: - Here $A = \begin{bmatrix} 2 & -4 & -1 \\ 0 & 3 & 4 \\ 1 & 6 & 2 \end{bmatrix}$

$\therefore$ the characteristic equation of the matrix A is [A-λI] =0

$$\begin{bmatrix} 2-\lambda & -4 & -1 \\ 0 & 3-\lambda & 4 \\ 1 & 6 & 2-\lambda \end{bmatrix} = 0$$

$\therefore (2-\lambda)[\,(3-\lambda)(\,2-\lambda)\text{-}24]+4[0\text{-}4]\text{-}1[0\text{-}(3-\lambda)] = 0$

$\therefore (2-\lambda)[\lambda^2 - 5\lambda + 6 - 24]\text{-}16+3-\lambda = 0$

$\therefore (2-\lambda)[\lambda^2 - 5\lambda - 18]\text{-}16+3-\lambda = 0$

$\therefore -\lambda^3 + 7\lambda^2 - 7\lambda - 49 = 0$

This is the characteristic equation of the given matrix.

$$\boldsymbol{\lambda^3 - D_1\lambda^2 + D_2\lambda - |A| = 0}$$

Where, D_1 = 2 +3 + 2 = 7 The diagonal element of matrix.

Where, D_2 = $\begin{vmatrix} 3 & 4 \\ 6 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$=(6-24)+(4+1)+6 = -7

And $|A| = \begin{vmatrix} 2 & -4 & -1 \\ 0 & 3 & 4 \\ 1 & 6 & 2 \end{vmatrix}$= 2(6-24)+4(0-4)-(0-3) =-36 -16+3 = -49

$$\boldsymbol{\lambda^3 - 7\lambda^2 - 7\lambda + 49 = 0}$$

Now, $A^2 = \begin{bmatrix} 2 & -4 & -1 \\ 0 & 3 & 4 \\ 1 & 6 & 2 \end{bmatrix} . \begin{bmatrix} 2 & -4 & -1 \\ 0 & 3 & 4 \\ 1 & 6 & 2 \end{bmatrix}$

$= \begin{bmatrix} 4+0-1 & -8-12-6 & -2-16-2 \\ 0+0+4 & 0+9+24 & 0+12+8 \\ 2+0+2 & -4+18+12 & -1+24+4 \end{bmatrix} = \begin{bmatrix} 3 & -26 & -20 \\ 4 & 33 & 20 \\ 4 & 26 & 27 \end{bmatrix}$

Now, $A^3 = A.\ A^2 = \begin{bmatrix} 2 & -4 & -1 \\ 0 & 3 & 4 \\ 1 & 6 & 2 \end{bmatrix} \begin{bmatrix} 3 & -26 & -20 \\ 4 & 33 & 20 \\ 4 & 26 & 27 \end{bmatrix}$

$= \begin{bmatrix} 6-16-4 & -52-132-26 & -40-80-27 \\ 0+12+16 & 0+99+104 & 0+60+108 \\ 3+24+8 & -26+198+52 & -20+120+54 \end{bmatrix} = \begin{bmatrix} -14 & -210 & -147 \\ 28 & 203 & 168 \\ 35 & 224 & 154 \end{bmatrix}$

Now, verify Caley Hamilton theorem

$\lambda^3 - 7\lambda^2 - 7\lambda + 49 = 0$ put $\lambda = A$ then we get,

$$A^3 - 7A^2 - 7A + 49I_3 = 0$$

$= \begin{bmatrix} -14 & -210 & -147 \\ 28 & 203 & 168 \\ 35 & 224 & 154 \end{bmatrix} - 7 \begin{bmatrix} 3 & -26 & -20 \\ 4 & 33 & 20 \\ 4 & 26 & 27 \end{bmatrix} - 7 \begin{bmatrix} 2 & -4 & -1 \\ 0 & 3 & 4 \\ 1 & 6 & 2 \end{bmatrix} + 49 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} -14 & -210 & -147 \\ 28 & 203 & 168 \\ 35 & 224 & 154 \end{bmatrix} + \begin{bmatrix} -21 & 182 & 140 \\ -28 & -231 & -140 \\ -28 & -182 & -189 \end{bmatrix}$

$+ \begin{bmatrix} -14 & 28 & 7 \\ 0 & -21 & -28 \\ -7 & -42 & -14 \end{bmatrix} + \begin{bmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{bmatrix}$

$= \begin{bmatrix} -14-21-14+49 & -210+182+28 & -147+140+7+0 \\ 28-28+0 & 203-231-21+49 & 168-140-28+0 \\ 35-28-7 & 224-182-42+0 & 154-189-14+49 \end{bmatrix}$

$$=\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

Thus, we get $A^3 - 7A^2 - 7A + 49I_3 = 0$

Therefore, Caley Hamilton theorem is verified.

Multiplying by A^{-1} to the equation $A^3 - 7A^2 - 7A + 49I_3 = 0$

Then,we get $A^2 - 7A - 7I_3 + 49A^{-1} = 0 \implies A^{-1} = \frac{1}{49}(-A^2 + 7A + 7I_3)$

$$=\frac{1}{49}\left(\begin{bmatrix}-3 & 26 & 20\\-4 & -33 & -20\\-4 & -26 & -27\end{bmatrix} + 7\begin{bmatrix}2 & -4 & -1\\0 & 3 & 4\\1 & 6 & 2\end{bmatrix} + 7\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right)$$

$$=\frac{1}{49}\left(\begin{bmatrix}-3+14+7 & 26-28+0 & 20-7+0\\-4+0+0 & -33+21+7 & -20+28+0\\-4+7+0 & -26+42+0 & -27+14+7\end{bmatrix}\right)$$

$$=\frac{1}{49}\left(\begin{bmatrix}18 & -2 & 13\\-4 & -5 & 8\\3 & 16 & -6\end{bmatrix}\right)$$

Check

$$\boldsymbol{AA^{-1} = I}$$

$$=\begin{bmatrix}2 & -4 & -1\\0 & 3 & 4\\1 & 6 & 2\end{bmatrix}\frac{1}{49}\left(\begin{bmatrix}18 & -2 & 13\\-4 & -5 & 8\\3 & 16 & -6\end{bmatrix}\right)$$

$$\boldsymbol{=\begin{bmatrix}\frac{36}{49}+\frac{16}{49}-\frac{3}{49} & \frac{-4}{49}+\frac{20}{49}-\frac{16}{49} & \frac{26}{49}-\frac{32}{49}+\frac{6}{49}\\ 0-\frac{12}{49}+\frac{12}{49} & 0+\frac{-15}{49}+\frac{64}{49} & 0+\frac{24}{49}-\frac{24}{49}\\ \frac{18}{49}-\frac{24}{49}+\frac{6}{49} & \frac{-2}{49}-\frac{30}{49}+\frac{32}{49} & \frac{13}{49}+\frac{48}{49}-\frac{12}{49}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}}$$

Example: Verify Calay-Hamilton theorem for the matrix $A = \begin{bmatrix}1 & 1 & 3\\1 & 3 & -3\\-2 & -4 & -4\end{bmatrix}$.

Also using this theorem find A^{-1}.

Solution: - Here $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

∴ the characteristic equation of the matrix A is [A-Iλ] =0

$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 3-\lambda & -3 \\ -2 & -4 & -4-\lambda \end{bmatrix} = 0$

∴ $(1-\lambda)[\,(3-\lambda)(-4-\lambda)-12]-1[(-4-\lambda)-6]+3[-4+2(3-\lambda)] = 0$

∴ $(1-\lambda)[\,(3-\lambda)(-4-\lambda)-12]-1[(-4-\lambda)-6]+3[-4+2(3-\lambda)] = 0$

∴ $(1-\lambda)[\lambda^2+\lambda-24]-1[(-10-\lambda)]+3[-2-2\lambda)] = 0$

∴ $-\lambda^3 + 20\lambda + 8 = 0$

This is the characteristic equation of the given matrix.

$$\boldsymbol{\lambda^3 - D_1\lambda^2 + D_2\lambda - |A| = 0}$$

Where, D_1 = 1 +3-4 = 0 The diagonal element of matrix.

Where, D_2 = $\begin{vmatrix} 3 & -3 \\ -4 & -4 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}$=(-12-12)+(-4+6)+(3-1) =-24+2+2=20

And $|A| = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{vmatrix}$= 1(-12-12)-1(-4-6)+3(-4+6) = -24 +10+6 = - 8

$$\boldsymbol{\lambda^3 + 20\lambda + 8 = 0}$$

Now, $A^2 = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} . \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

$$= \begin{bmatrix} 1+1-6 & 1+3-12 & 3-3-12 \\ 1+3+6 & 1+9+12 & 3-9+12 \\ -2-4+8 & -2-12+16 & -6+12+16 \end{bmatrix} = \begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix}$$

Now, $A^3 = A.\ A^2 = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}\begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix}$

$$= \begin{bmatrix} -4+10+6 & -8+22+6 & -12+6+66 \\ -4+30-6 & -8+66-6 & -12+18-66 \\ 8-40-8 & 16-88-8 & 24-24-88 \end{bmatrix} = \begin{bmatrix} 12 & -20 & 60 \\ 20 & 52 & -60 \\ -40 & -80 & -88 \end{bmatrix}$$

Now, verify Caley Hamilton theorem

$\lambda^3 + 20\lambda + 8 = 0$ put λ =A then we get,

$$A^3 - 20A + 8I_3 = 0$$

$$= \begin{bmatrix} 12 & -20 & 60 \\ 20 & 52 & -60 \\ -40 & -80 & -88 \end{bmatrix} - 20\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & -20 & 60 \\ 20 & 52 & -60 \\ -40 & -80 & -88 \end{bmatrix} + \begin{bmatrix} -20 & -20 & -60 \\ -20 & -60 & 60 \\ 40 & 80 & 80 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 12-20+8 & -20-20+0 & 60-60+0 \\ 20-20+0 & 52-60+8 & -60+60+0 \\ -40+40+0 & -80+80+0 & -88+80+8 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, we get $A^3 - 20A + 8I_3 = 0$

Therefore, Caley Hamilton theorem is verified.

Multiplying by A^{-1} to the equation $A^3 - 20A + 8I_3 = 0$.

Then, we get $A^2 - 20I_3 + 8A^{-1} = 0$

$$A^{-1} = \frac{1}{8}(-A^2 + 20I_3) = \frac{1}{8}\left(\begin{bmatrix} 4 & 8 & 12 \\ -10 & -22 & -6 \\ -2 & -2 & -22 \end{bmatrix} + 20\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= \frac{1}{8}\begin{bmatrix} 24 & 8 & 12 \\ -10 & -2 & -6 \\ -2 & -2 & -2 \end{bmatrix}$$

Check

$$AA^{-1} = I$$

$$= \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} \frac{1}{8}\begin{bmatrix} 24 & 8 & 12 \\ -10 & -2 & -6 \\ -2 & -2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 - \frac{5}{4} - \frac{3}{4} & 1 - \frac{1}{4} - \frac{3}{4} & \frac{3}{2} - \frac{3}{4} - \frac{3}{4} \\ 3 - \frac{15}{4} + \frac{3}{4} & 1 - \frac{3}{4} + \frac{3}{4} & \frac{3}{2} - \frac{9}{4} + \frac{3}{4} \\ -6 + 5 + 1 & -2 + 1 + 1 & -3 + 3 + 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Simultaneous linear equations and their solution:

Definition:- linear equations

If $a_1, a_2, a_3, \dots a_n$ and b are real numbers and $x_1, x_2, x_3, \dots x_n$ are variables, then $a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$ is called a linear equation in n variables $x_1, x_2, x_3, \dots x_n$.

Note: consider a system of m linear equations in n unknowns $x_1, x_2, x_3, \dots x_n$ in the following form:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$

……… ……. …….. ……. …… ……

……… ……. …….. ……. …… ……

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

Here $b_i, a_{ij} \in R$ $(i = 1,2, \dots, m; j = 1,2,3, \dots, n)$ are the fixed numbers. If we get $x = (k_{1,} k_{2,} k_{3,} \dots, k_n) \in R^n$ such that $x_1 = k_{1,} x_2 = k_{2,} x_3 = k_{3,} \dots x_n = k_n$ satisfy above the system of equations then *x* is called a solution of the above system of equations. The set of all possible solutions of above system of equations is called **the solution set** above system of equations.

Above system of equations can be linearly written as $\sum_{j=1}^{n} a_{ij}x_j = b_i, i = 1,2,3, \dots, m$. If we put A = $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, X = $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and B = $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ then

Theabove system of equations can be expressed in the matrix form A X = B.

Here A is called the coefficient matrix of the above system of equations.

If $b_1 = b_2 = b_3 = \cdots = b_m = 0$, then the system of above equations is called **Homogeneous system**.

Definition: - Homogeneous systemof equations

Consider a system of m linear equations in n unknowns $x_1, x_2, x_3, \dots x_n$ in the following form:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = 0$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = 0$$

……… ……. …….. ……. …… ……

……… ……. …….. ……. …… ……

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = 0$$

is called **Homogeneous system of equations**.

Note: If $x_1 = 0, x_2 = 0, x_3 = 0 \ldots x_n = 0$ is a solution of homogeneous system of equations then it is called **trivial solution** of the system and any other solution is called **non-trivial** solution.

Definition: -augmented matrix.

For the above system A X = B, the matrix $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$ is called augmented matrix and is denoted by the symbol [A,B]

Theorem: -The nonhomogeneous system as $\sum_{j=1}^{n} a_{ij}x_j = b_i, i = 1,2,3, \ldots, m.$of linear equations has a solution if and only if the ranks of coefficient matrix and augmented matrix are equal.

Proof: -Let us suppose that the solution of the given system exists and let it be $(k_{1,}k_{2,}k_{3,} \ldots, k_n) \in R^n$.

The given system can be expressed as

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

If $(k_{1,}k_{2,}k_{3,} \ldots, k_n)$ is the solution of the given system then $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ can be expressed as a linear combination of the column matrices of A = $[a_{ij}]_n$.

Therefore, the number of linearly independent column matrices of A and [A, B] is same.

Hence, ranks of A and [A, B] are equal.

Conversely,

Let us suppose that ranks of A and [A, B] are equal.

Thus, $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ must be a linear combination of the column matrices of A.

Therefore, their exists real numbersα_1, α_2,α_3, ….,α_n such that

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \alpha_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + \alpha_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + \alpha_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Therefore, (α_1, α_2,α_3, ….,α_n) $\in R^n$ is the solution of the given system.

Theorem: -If s =$\left(s_{1,}s_{2,}s_{3,} \dots , s_n\right) \in R^n$is a particular solution of the nonhomogeneous system of equation$\sum_{j=1}^{n} a_{ij}x_j = b_i, i = 1,2,3, \dots , m.$ and x = $\left(k_{1,}k_{2,}k_{3,} \dots , k_n\right) \in R^n$ is a solution of the homogeneous system of equation $\sum_{j=1}^{n} a_{ij}x_j = 0 \; i = 1,2,3, \dots , m.$then s + x is a solution of the system $\sum_{j=1}^{n} a_{ij}x_j = b_i, i = 1,2,3, \dots , m.$ Moreover each solution of this system is of the form s + x .

Proof: -System of equations $\sum_{j=1}^{n} a_{ij}x_j = b_i, i = 1,2,3, \dots , m.$ ______(1)

and $\sum_{j=1}^{n} a_{ij}x_j = 0 \; i = 1,2,3, \dots , m.$ ____________(2)

Now, for s +x=$\left(s_1 + k_{1,}s_2 + k_{2,}s_3 + k_{3,} \dots , s_n + k_n\right)$

$$\sum_{j=1}^{n} a_{ij}(s_j + k_j) = \sum_{j=1}^{n} a_{ij}s_j + \sum_{j=1}^{n} a_{ij}k_j = b_i, i = 1,2,3, \dots , m.$$

Because $\sum_{j=1}^{n} a_{ij}x_j = b_i, i = 1,2,3, \dots , m.$ and $\sum_{j=1}^{n} a_{ij}x_j = 0 \; i = 1,2,3, \dots , m.$

Let us suppose that y =$\left(l_{1,}l_{2,}l_{3,} \dots , l_n\right)$ is the any solution of the given system (1). Then y- s = $(l_1 - s_1 l_2 - -s_2 l_3 - s_3 \dots , l_n - s_n)$

Therefore, $\sum_{j=1}^{n} a_{ij}(l_j - s_j) = \sum_{j=1}^{n} a_{ij}l_j - \sum_{j=1}^{n} a_{ij}s_j = b_i - b_i = 0, \; i = 1,2,3, \dots , m.$

Therefore y-s is a solution of the homogeneous system (2).

Put y -s = x.

Thus, every solution of system (1) if equations is of the form s +x.

Theorem: -Let$\sum_{j=1}^{n} a_{ij} - x_i = 0(i = 1,2,3, \dots, n)$ be system of n equations in n unknowns. If the coefficient matrix of the system is singular (not invertible), then and only then the system has a non-trivial solution.

Proof: -Let A =$[a_{ij}]_n$ is a coefficient matrix of the given system. The given system can be expressed in the form

$x_1 c_1 + x_2 c_2 + \cdots + x_n c_n = 0$ …………..(1)

Where $c_1, c_2, \dots, c_n$ are column matrices of A.

If the given system has a non-trivial solution, then there exists $i \in \{1,2,3, \dots, n\}$ such that $x_i \neq 0$. It is clear from result (1) that the column matrices of the matrix A are linearly dependent.

Therefore, the rank of A is less than n.

i.e. $|A| = 0$ or A is singular.

Conversely,

If A is singular, then r(A)<n.

Therefore, the column matrices of A are linearly dependent. Consequently some x_i is non-zero.

Therefore, the given system has a non-trivial solution.

Theorem: - Let $\sum_{j=1}^{n} a_{ij} x_j = b_i, i = 1,2,3, \dots, n$.be system of n equations in n unknowns. This system has a unique solution if and only if the coefficient matrix is invertible. (i. e. it has an inverse.)

Proof: -Let A =$[a_{ij}]_n$ is a coefficient matrix of the given system.

If A is invertible, then r(A) = n.

Also, from $\sum_{j=1}^{n} a_{ij}x_j = b_i, i = 1,2,3, \dots, n$. It is clear that $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ is a linear combination of the column matrices of A.

∴ r[A] = r [A, B], where [A, B] is the augmented matrix of the system.

∴ the system has a solution.

Let $x = (k_1, k_2, k_3, \dots, k_n) \in R^n$ be solution of the system. The homogeneous system corresponding to the given system is

$\sum_{j=1}^{n} a_{ij}x_j = 0 \; i = 1,2,3, \dots, n.$ …………………(1)

A is invertible.

Hence, the system (1) cannot have a non-trivial solution.

∴ $x_1 = 0, x_2 = 0, x_3 = 0, \dots, x_n = 0$ is the solution of the system (1).

So, given system has a unique solution $x + 0 = x$.

Conversely,

Let us suppose that the given system has unique solution. Consequently, the corresponding homogeneous system has only trivial solution. Thus, the coefficient matrix A is invertible.

Crammer's Rule:-Let A $=[a_{ij}]_n$ be square matrix with $|A| \neq 0$. Let B = $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ be column vector. Then the solution of AX = B is given by $x_j = \frac{|A_1,\dots,B,\dots,A_n|}{|A|}$ 0r $x_j = \frac{\det(A_1,\dots,B,\dots,A_n)}{\det(A)}$ Where, B is in the j^{th} place.

i.e.

Consider the system AX = B of n linear equations in n unknowns, where A $=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, X = $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and B = $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$.

If $|A| \neq 0$, thenA^{-1} exists.

Now AX = B ⇒X = A^{-1}B.

But $A^{-1} = \frac{adj\text{ A}}{|A|}$

$\therefore X = \frac{adj\text{ A}}{|A|}B. = \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$. Where$A_{ij}$ is the cofactor of a_{ij} in $|A|$

Thus, $x_1 = \frac{|A_1|}{|A|}, x_2 = \frac{|A_2|}{|A|}, \ldots, x_i = \frac{|A_i|}{|A|}, \ldots, x_n = \frac{|A_n|}{|A|}$ Where A_i is the matrix obtained from A by replacing the i^{th} column by constant column $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$. This method of solving n equations is known as crammer's rule.

Note: The system of linear equations is called consistent if it has a solution. If it does not have any solution, then it is called in consistent.

Example: -Solve $5x + 3y + 7z = 4; 3x + 26y + 2z = 9; 7x + 2y + 11z = 5$ using Crammer's rule.

Solution: - Here A $=\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 11 \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and B = $\begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$

$|A| = \begin{vmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 11 \end{vmatrix}$=5(286-4)-3(33-14)+7(6-182) =1410-57-1232=121

$$x = \frac{\begin{vmatrix} 4 & 3 & 7 \\ 9 & 26 & 2 \\ 5 & 2 & 11 \end{vmatrix}}{|A|} = \frac{77}{121} y = \frac{\begin{vmatrix} 5 & 4 & 7 \\ 3 & 9 & 2 \\ 7 & 5 & 11 \end{vmatrix}}{|A|} = \frac{33}{121} \text{ and } z = \frac{\begin{vmatrix} 5 & 3 & 4 \\ 3 & 26 & 9 \\ 7 & 2 & 5 \end{vmatrix}}{|A|} = \frac{0}{121}$$

Example: -Solve x +y = 0; y + z = 1;x +z = -1 using Crammer's rule.

Solution: - Here A = $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$and B = $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$=1(1-0)-1(0-1)+0(0-1) =2

$$x = \frac{\begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 1 \end{vmatrix}}{|A|} = \frac{-2}{2} = -1 y = \frac{\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix}}{|A|} = \frac{2}{2} = 1 \text{ and } z = \frac{\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix}}{|A|} = \frac{0}{2} = 0$$

Example: -Solve 2x +y = 0;3y + z = 1;x +4z = 2 using Crammer's rule.

Solution: - Here A = $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 1 & 0 & 4 \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$and B = $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

$|A| = \begin{vmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 1 & 0 & 4 \end{vmatrix}$=2(12-0)-1(0-1)+0(0-3) =24+1+0=25

$$x = \frac{\begin{vmatrix} 0 & 1 & 0 \\ 1 & 3 & 1 \\ 2 & 0 & 4 \end{vmatrix}}{|A|} = \frac{-2}{25} y = \frac{\begin{vmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix}}{|A|} = \frac{4}{25} \text{ and } z = \frac{\begin{vmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 1 & 0 & 2 \end{vmatrix}}{|A|} = \frac{13}{25}$$

$x = \frac{-2}{25} y = \frac{4}{25}$ and $z = \frac{13}{25}$

Example: -Solve the following system of equations. **OR** Prove that following system of equations is consistent.

2x +5y +6z = 13;3x +y -4z = 0;x -3y -8z = -10.

Solution: -Augmented matrix is [A, B]= $\begin{bmatrix} 2 & 5 & 6 & 13 \\ 3 & 1 & -4 & 0 \\ 1 & -3 & -8 & -10 \end{bmatrix}$

$R_1 \leftrightarrow R_3$. $\sim \begin{bmatrix} 1 & -3 & -8 & -10 \\ 3 & 1 & -4 & 0 \\ 2 & 5 & 6 & 13 \end{bmatrix}$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1 \sim \begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 10 & 20 & 30 \\ 0 & 11 & 22 & 33 \end{bmatrix}$

$R_2 \to \frac{1}{10}R_2$, $R_3 \to \frac{1}{11}R_3$then $R_3 \to R_3 - R_2 \sim \begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

r(A, B) = r(A)= 2<3

So, the given system is consistent. Solution is not unique. i.e., system has infinite solution.

Also, the given system of equations is equivalent to

x -3y -8z = -10 ---------(1)

y +2z = 3 ------------(2)

from (2) we get y = 3-2z, so, (1) gives x=-10+9-6z+8z = -1+2z

so, the solution is -1 +2k, 3-2k, k where k ∈R.

Thus, set of all solution is

=(-1 +2k, 3-2k, k)/k ∈R} = {(-1,3,0) +k(2, -2, 1)/k ∈R}

Example: -Solve the following system of equations

5x +3y +7z = 4;3x +26y +2z = 9;7x +2y +11z = 5.

Solution

Example: -Solve the following system of equations. **OR** Prove that following system of equations is consistent.2x +y = 0;3y + z = 1;x +4z = 2.

Solution: - Here Augmented matrix is [A, B] = $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 4 & 2 \end{bmatrix}$

$R_1 \leftrightarrow R_3.\sim \begin{bmatrix} 1 & 0 & 4 & 2 \\ 0 & 3 & 1 & 1 \\ 2 & 1 & 0 & 0 \end{bmatrix}$

$R_3 \to R_3 - 2R_1 \sim \begin{bmatrix} 1 & 0 & 4 & 2 \\ 0 & 3 & 1 & 1 \\ 0 & 1 & -8 & -4 \end{bmatrix}$ $R_2 \leftrightarrow R_3.\sim \begin{bmatrix} 1 & 0 & 4 & 2 \\ 0 & 1 & -8 & -4 \\ 0 & 3 & 1 & 1 \end{bmatrix}$

$R_3 \to R_3 - 3R_2 \sim \begin{bmatrix} 1 & 0 & 4 & 2 \\ 0 & 1 & -8 & -4 \\ 0 & 0 & 25 & 13 \end{bmatrix}$ $R_3 \to \frac{1}{25} R_3 \sim \begin{bmatrix} 1 & 0 & 4 & 2 \\ 0 & 1 & -8 & -4 \\ 0 & 0 & 1 & \frac{13}{25} \end{bmatrix}$

$R_2 \to R_2 + 8R_3,\ R_1 \to R_1 - 4R_3 \sim \begin{bmatrix} 1 & 0 & 0 & \frac{-2}{25} \\ 0 & 1 & 0 & \frac{4}{25} \\ 0 & 0 & 1 & \frac{13}{25} \end{bmatrix}$

r(A, B) = r(A)= 3

So, the given system is consistent and has unique solution

$x = \frac{-2}{25}\, y = \frac{4}{25}$ and $z = \frac{13}{25}$ is the unique solution of the given equations.

Example: -Solve the following system of equations. **OR** Prove that following system of equations is consistent. $2x + 6y = 15; 6x + 20y - 6z = 2; 6y - 18z = 7$.

Solution: - Here Augmented matrix is [A, B] = $\begin{bmatrix} 2 & 6 & 0 & 15 \\ 6 & 20 & -6 & 2 \\ 0 & 6 & -18 & 7 \end{bmatrix}$

$R_2 \to R_2 - 3R_1.\sim \begin{bmatrix} 2 & 6 & 0 & 15 \\ 0 & 2 & -6 & -43 \\ 0 & 6 & -18 & 7 \end{bmatrix}$

$R_3 \to R_3 - 3R_2.\sim \begin{bmatrix} 2 & 6 & 0 & 15 \\ 0 & 2 & -6 & -43 \\ 0 & 0 & 0 & 129 \end{bmatrix}$

So, we get **r(A, B) =3 and r(A)= 2**

Therefore, r(A, B) ≠r(A)

So, the given system is inconsistent. So, we cannot find the solution of given system of equations.

Example: -Solve the following system of equations. **OR** Find the value of μ if following system of equations is consistent. $x+2y+3z=14$; $x+4y+7z=30$; $x+y+z=\mu$.

Solution: - Here Augmented matrix is [A, B] = $\begin{bmatrix} 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \\ 1 & 1 & 1 & \mu \end{bmatrix}$

$R_3 \to R_3 - R_1$ $R_2 \to R_2 - R_1$.~ $\begin{bmatrix} 1 & 2 & 3 & 14 \\ 0 & 2 & 4 & 16 \\ 0 & -1 & -2 & \mu - 14 \end{bmatrix}$

$R_2 \to \frac{1}{2} R_2$.~ $\begin{bmatrix} 1 & 2 & 3 & 14 \\ 0 & 1 & 2 & 8 \\ 0 & -1 & -2 & \mu - 14 \end{bmatrix}$ $R_3 \to R_3 + R_2$.~ $\begin{bmatrix} 1 & 2 & 3 & 14 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & \mu - 6 \end{bmatrix}$

If $\mu \neq 6$ then r(A, B) = 3 and r(A) = 2

i.e. r(A, B)≠ r(A) and system is inconsistent.

While if μ= 6 then (A, B) =r(A) =2<3.

So, the system will be consistent and have infinite solutions.

Also, $x+2y+3z=14$;

$y+2z=8 \Rightarrow y=8-2z$ and $x=-2+z$

So if μ= 6 then only the given system is consistent and solution is

(-2 +k, 8-2k, k)/k ∈R} = {(-2, 8, 0) +k(1, -2, 1)/k ∈R}