Sem-III
MAT 202: Linear Algebra-I
Unit-1 Vector space

Definition:- Vector space:

Let V be a set on which addition and scalar multiplication are defined (this means that if **u** and **v** are objects in V and c is a scalar then we've defined $\mathbf{u}+\mathbf{v}$ and c**u** in some way). If the following axioms are true for all objects **u**, **v**, and **w** in V and all scalars c and k then V is called a **vector space** and the objects in V are called **vectors**.

(a) u + v is in V — This is called **closed under addition**.
(b) c**u** is in V — This is called **closed under scalar multiplication**.
(c) u + v = v + u
(d) u +(v + w) = (u + v) + w
(e) There is a special object in V, denoted **0** and called the **zero vector**, such that for all **u** in V we have u + 0 = 0 + u = u

(f) For every **u** in V there is another object in V, denoted -u and called the **negative** of **u**, such that u – u = u +(-u) = 0 .
(g) c (**u + v) = cu + cv**
(h) (c + k) u = cu + ku
(i) c (ku) = (ck)u
(j) 1u = u

Remark: A complex vector space is defined as above by using complex numbers instead of real numbers.

Theorem:- Suppose that V is a vector space, **u** is a vector in V and α is any scalar. Then,
(a) 0u = 0
(b) $\alpha\, 0 = 0$
(c) (-1u) =-u

Proof :
(a) 0u=(0 +0)u = 0u +0u
Adding –(0u) to both sides, we get
0u +(-0u) = 0u +0u +(-0u)
0 = 0u + 0 ($\because$ 0 is Additive identity)
0 = 0u

(b) $\alpha\, 0 = \alpha(0+0)$ ($\because$ 0 is Additive identity)
$= \alpha\, 0 + \alpha\, 0$

Adding $-(\alpha 0)$ to both sides, we get

$0 = -(\alpha 0) + (\alpha 0 + \alpha 0)$

$= (-(\alpha 0) + (\alpha 0)) + \alpha 0$ ($\because$ Associative law)

$= 0 + \alpha 0$ ($\because -(\alpha 0)$ is Additive inverse)

$= \alpha 0$ ($\because 0$ is Additive identity)

(c) In this case if we can show that $u + (-1)u = 0$ then from axiom **(f)** we'll know that $(-1)u$ is the negative of **u**, or in other words that $(-1)u = -u$.

$u + (-1)u = 1.\ u + (-1)u$ ($\because 1$ is multiplicative identity)

$= (1 + -1)\, u$ ($\because$ -1 is Additive inverse of 1)

$= 0u$

$= 0$

So by uniqueness of the negative, (-1)u is the negative of u, i.e. (-1u) = -u

Example:-

Definition:- Sub spaces:-

Let S be a non empty subset of a vector space V. S is said to be a subspace of V if S is a vector space under the same operations of addition and scalar multiplication as in V.

Geometric meaning of vector sub space:

Question :- Prove that every line through the origin is a subspace of $\mathbf{V_2}$.

In Euclidean space $\mathbf{V_2}$, take any straight line S through the origin O. Any point P on this straight line can be considered as a vector $\overrightarrow{op}$ of $\mathbf{V_2}$ in S. the sum of two such vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, where P and Q both lie in S. is again a vector $\overrightarrow{OR}$, where R lies in S.

Similarly, a scalar multiplication of any vector in S is again a vector in S. All other axioms are automatically satisfied in S. So S is a vector space under the same operations as in $\mathbf{V_2}$. Thus S is a subspace of $\mathbf{V_2}$. i.e. every line through the origin is a subspace of V_2.

Same way, in V_3 we can find that any plane S through the origin is a subspace of V_3. Also every line L through the origin is a subspace of V_3.

Theorem:- A non empty subset S of a vector space V is a subspace of V iff the following conditions are satisfied:

(a) If $u, v \in S$ then $u + v \in S$.

(b) If $u \in S$ and α a scalar, then $\alpha u \in S$.

(OR)

A subset S of a vector space V is a subspace of V iff it is closed under addition and scalar multiplication defined in V.

Proof: Let S be subspace of V is given.

We shall prove that condition (a) and (b) hold in S.

Since S is subspace of V.

$\therefore$ it is closed under addition and scalar multiplication defined in V.

($\because$ It is a vector space itself.)

Hence the result.

Conversively,

Let S is closed under addition and scalar multiplication is given.

We shall prove that S is a vector sub space of V.

(I) Since S is closed under scalar multiplication is given.

$\therefore \quad \forall u \in S, -1 \in V \Rightarrow (-1)u \in S$

$$\Rightarrow -u \in S$$

$$[\because u \in S \Rightarrow -u \in V and\, (-1)(u) = -u \in V\,]$$

Thus additive inverse of each element of S exists.

(II) Since S is closed under vector addition is given.

$\forall u \in S, -u \in S \Rightarrow u + (-u) \in S$

$$[\because u \in S \Rightarrow -u \in S \Rightarrow u + (-u) = 0 \in V]$$

$$\Rightarrow 0 \in S$$

Thus 0 is the additive identity of S.

(III) Since elements of S are elements of V,

$\therefore$ vector addition is commutative and associative in S.

Thus, S is an abelian group under vector addition.

Further S is closed under scalar vector multiplication and therefore, the remaining properties of vector space also hold in S because they hold in V.

Example:-

Let L be the set of all vectors of the form $(x, 2x, -3x, x)$ in $V_{4.}$ Then L is a subspace of $V_{4.}$

Sol^{n}:-

Let $u = (x, 2x, -3x, x)$ and $v = (y, 2y, -3y, y)$.

Here $u, v \in L$.

Now,

$$u + v = (x, 2x, -3x, x) + (y, 2y, -3y, y)$$
$$= [x + y, 2(x + y), -3(x + y), x + y]$$
$$= (z, 2z, -3z, z) \in L.$$

Where $z = x + y$.

$\therefore u + v \in L.$

Similarly if $\alpha \in R$ then

$$\alpha u = \alpha(x, 2x, -3x, x)$$
$$= [\alpha x, 2(\alpha x), -3(\alpha x), \alpha x] \in L.$$

$\therefore \alpha u \in L.$

Hence L is subspace of $V_{4.}$

Example:-

The set S of all polynomials $P \in \mathfrak{p}$, which vanishes at a fixed point x_0, is a subspace of $\mathfrak{p}$.

Sol^n:-

We have $S = \{p \in \mathfrak{p} / P(x_0) = 0\}$.

Let $p,q \in S$. such that $P(x_0)=0$ and $q(x_0)=0$.

Now,

$(p+q)(x_0) = p(x_0) + q(x_0) = 0$.

$\therefore (p+q) \in S$.

i.e. The polynomial $p+q$ also vanished at x_0.

So, addition is closed in S.

Similarly, if $\alpha \in R$ and $P \in S$ then

$\alpha p(x_0) = \alpha(p(x_0)) = \alpha \cdot 0 = 0.$

$\therefore \alpha p \in S$.

So scalar multiplication in S.

$\therefore$ S is subspace of $\mathfrak{p}$.

Note :-

- The set containing just the zero element of and nothing else is a subspace of V. i.e.{0} is subspace of V.
- Also vector space V is itself subspace.
- Subspace {0} and V of V are called trivial subspaces of V and all other subspaces of v are called nontrivial subspaces of V.
- The trivial subspace {0} of V is denoted by V_0 and is also called the zero subspace of V.

Example:- Prove that s= $\{(x_1, x_2, x_3, \ldots x_n) / (x_1, x_2, x_3, \ldots x_n) \in V_n\}$ is vector subspace of V_n (OR)

Prove that the equation $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = 0$ ---(1) where α_i's are real constants and x_i's are real unknowns is vector subspace of V_n

Sol^n:-

A solution of this equation can be represented as an n-tuple $(x_1, x_2, x_3, \ldots x_n)$ which is a vector of V_n.

Let s be the set of all vector $(x_1, x_2, x_3, \ldots x_n) \in V_n$

i.e. s= $\{(x_1, x_2, x_3, \ldots x_n) / (x_1, x_2, x_3, \ldots x_n) \in V_n\}$ which satisfy the equation (1).

$\therefore$ S is a subspace of V_n.

Because

Let $x, y \in S$ such that

$x = (x_1, x_2, \ldots, x_n)$ $\quad$ $y = (y_1, y_2, \ldots, y_n)$ then

$x + y = (x_1 + y_1, x_2 + y_2, \ldots x_n + y_n)$

Which is also solution of equation (1).

$\therefore\ x+y\in S$

Similarly $\forall \alpha\in R$

$\alpha x\in S$

$\therefore$ S is a subspace of $V_{n.}$

Note :-

- The set S of vectors $(x, y) \in V_2$ which satisfy the equation $\alpha x+\beta y=0$ is clearly a straight line through the origin in V_2.
  $\therefore$ S is subspace of V_2.
- In V_3 the set of all vectors $(x, y, z) \in V_3$,which satisfy the equation $\alpha x+\beta y+\gamma z=0$ is a plane through the origin and hence a subspace of V_3. But in V_3 we consider the set S of all vectors (x,y,z) which satisfy the equation $\alpha x+\beta y+\gamma z=1$ is a plane, but it does not contain the vector(0,0,0).So It is not a subspace.

Problem set 3.2

Example:1 Prove that a subset W of a vector space V is subspace of V iff

$\alpha x+\beta y\in W \ \ \forall \alpha,\beta\in R$ and $\forall x,y\in W$.

Solution:- Let us suppose that W is a subspace of V.

We have to prove that $\alpha x+\beta y\in W \ \ \forall \alpha,\beta\in R$ and $\forall x,y\in W$.

Since W is a subspace of V.

$\alpha x\in W \ \ \forall \alpha\in R$ and $\forall x\in W$.

Also $\beta y\in W \ \ \forall \beta\in R$ and $\forall y\in W$.

Since W is closed under scalar multiplication.

Hence, $\alpha x+\beta y\in W \ \ \forall \alpha,\beta\in R$ and $\forall x,y\in W$.

Convesely,

Let $\alpha x+\beta y\in W \ \ \forall \alpha,\beta\in R$ and $\forall x,y\in W$ is given.

Now we want to prove that W is a subspace of V.

Let us take $\alpha=1$ and $\beta=-1$ then $x-y\in W$

Let us take $\beta=0$ then $\alpha x+0y\in W\Rightarrow \alpha x\in W$

Hence W is a subspace of V.

Span of a Set

Definition:- Linear combination:-

Let $u_1, u_2, \ldots, u_n$ be n vectors of a vector space V and $\alpha_1, \alpha_2, \ldots, \alpha_n$ be n scalars. Then $\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$ is called a linear combination of $u_1, u_2, \ldots, u_n$.

It is also called a linear combination of the set $\{ u_1, u_2, \ldots, u_n \}$

If a linear combination of a finite set then it is also called a finite linear combination.

Definition:- Span of a set:-

The span of a subset S of a vector space V is the set of all finite linear combination of S.

(OR)

If S is a subset of V, the span of S is the set
$\{ \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n / \alpha_i \in R, u_i \in S, 1 \le i \le n, n \in N \}$

The span of S is denoted by [S].

If S contains only a finite number of elements, say $u_1, u_2, \ldots, u_n$, then [S] is also written as $[u_1, u_2, \ldots, u_n]$.

e.g. Let us take V = V_3 and S = { (1,0,0) , (0,1,0) } then the linear combination of set S as $\alpha(1,0,0) + \beta(0,1,0) = (\alpha, \beta, 0)$

The set of all such linear combination is [s]. i.e. span of set S.

i.e. [s] = $\{ (\alpha, \beta, 0) / \alpha, \beta \in R \}$ or [S] = [(1,0,0) , (0,1,0)]

Theorem:- Let S be a nonempty subset of a vector space V, then prove that [s] is a subspace of V.

[OR]

Let S be a nonempty subset of a vector space V, then prove that the span of S is a subspace of V.

Proof:- We have to prove that [S] is a subspace of V.

For this,

Let u, v $\in$ [S] such that

$u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$ and $v = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_m v_m$

For some scalars α_i and β_i and $u_i's, v_i's \in S \quad \forall n \in N$

$\therefore u + v = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n + \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_m v_m$

Here $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_m \in$ S and $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_m$ are scalars.

$\therefore u + v \in [S]$ ____________(i)

Similarly,

$\alpha u = (\alpha\alpha_1)u_1 + (\alpha\alpha_2)u_2 + \cdots + (\alpha\alpha_n)u_n$ is again a finite linear combination of S.

$\therefore \alpha u \in [S]$ __________________(ii)

From (i) and (ii)

[S] is a subspace of V.

Note:- A nontrivial subspace always contains an infinite number of elements. So [S] ($\neq V_0$) always contains an infinite numbers. But S itself may be a smaller set, even a finite set. By convention we take $[\phi] = V_0$

Theorem:- If S is a nonempty subset of a vector space, then prove that [S] is the smallest subspace of V containing S.

Proof:- We know that [S] is a subspace of V.

Since s $\in$ [S]

Because each elements u_0 of S can be written as $1.u_0$.

i.e. $1.u_0 \in$ [S]

now we want to prove that [S] is the smallest subspace containing S.

For this,

We shall show that if there exist another subspace T containing S, then T contains [S] also.

Let a subspace T contain S.

i.e. S $\subset$ T.

Let us take any element u $\in$ [S]

Where $u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$

For some scalars α_i and $u_i's \in S \quad \forall n \in N$

since S $\subset$ T.

$\therefore u_1, u_2, \ldots, u_n \in$ T

Since T is a subspace.

$\therefore \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n \in$ T

i.e. u $\in$ T

$\therefore$ [S] $\subset$ T

This prove that [S] is the smallest subspace of V containing S.

Example:- In V_2 show that (3, 7) belongs to [(1, 2), (0, 1)] but does not belongs to [(1, 2), (2, 4)].

Solution:- (3, 7) $\in$ [(1, 2), (0, 1)]

if (3, 7) is linear combination of (1, 2) and (0, 1) .

i.e. α, β be scalars such that (3, 7) = α (1, 2) + β (0, 1)

$\therefore \alpha = 3$ and $2\alpha + \beta = 7$

Solving these equation then we get

$\alpha = 3$ and $\beta = 1$

Thus, (3, 7) =3 (1, 2) +1. (0, 1)

Hence (3, 7) $\in$ [(1, 2), (0, 1)]

Now, if (3, 7) $\in$ [(1, 2), (2, 4)]

if (3, 7) is linear combination of (1, 2) and (2, 4) .

i.e. α, β be scalars such that (3, 7) $= \alpha$ (1, 2) $+ \beta$ (2, 4)

$\therefore \quad \alpha + 2\beta = 3$ and $2\alpha + 4\beta = 7$

But these equation can not hold at same time

Becuse $2\alpha + 4\beta = 6$

$2\alpha + 4\beta = 7$

This is not possible.

(3, 7) $\notin$ [(1, 2), (2, 4)]

Example:-In the complex vector space V_2^c . Show that (1 + i , 1- i) belongs to [(1 + i , 1), (1, 1- i)]

Solution:- α, β be scalars such that (1 + i , 1- i) $=\alpha$ (1 + i , 1)+ β (1, 1- i)

$\therefore$ 1 + i $=\alpha (1 + i) + \beta$ and 1- i = $\alpha + \beta(1- i)$

Solving these equation then we get

$\alpha = 1 + i$ and $\beta = 1- i$

$\therefore$ (1 + i , 1- i) $\in$ [(1 + i , 1), (1, 1- i)]

Example:- If U and W is subspaces of V then prove that U$\cap$ W is subspace of V.

Solution:- We want to prove that U$\cap$ W is subspace of V.

For this,

Let $u, v \in$ U$\cap$ W

$\therefore u, v \in$ U and $u, v \in$ W

Since U and W is subspaces of V.

$\therefore u + v \in$ U and $u + v \in$ W

$\therefore u + v \in$ U $\cap$ W.

Similarly,

$\forall \alpha \in R,\ \alpha u \in$ U $\cap$ W

(because U and W is subspaces of V $\therefore \alpha u \in$ U and $\alpha u \in$ W)

$\therefore$ U$\cap$ W is subspace of V.

Note:- (1) If $U_1, U_2, U_3, \ldots, U_n$ are n subspaces of V then their intersection $U_1 \cap U_2 \cap U_3 \cap \ldots \cap U_n$ is also a subspace of V.

(2) Let U and W be subspaces of a vector space V. then their intersection U $\cap$ W cannot be empty because each contains the zero vector of V.

Example:- Let W be the set of all vectors $(x_1, x_2, x_3, \ldots, x_n)$ of Vn satisfying the three equations.

$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \cdots + \alpha_n x_n = 0$ ----------------(1)

$\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \cdots + \beta_n x_n = 0$ ---------------(2)

$\gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_3 + \cdots + \gamma_n x_n = 0$ ---------------(3)

Then W = $W_1 \cap W_2 \cap W_3$

Where W_1 is the solution set of equation (1)

W_2 is the solution set of equation (2)

W_3 is the solution set of equation (3)

Solution:- Since each W_i is subspaces.

$\therefore$ W is subspace of V_n.

Example:- Prove that the union of two subspaces of V need not be a subspace of V.

Solution:- Let us take U = x-axis and W = Y-axis in V_2.

Here U and W are subspaces of V_2.

Here (1, 0) $\in$ U and (0, 1) $\in$ W

So (1, 0) and (0, 1) $\in U \cup W$

But (1, 0) + (0, 1) =(1, 1) $\notin U \cup W$ ($\because$ (1, 1) $\notin$ U and (1, 1) $\notin$ W)

This show that $U \cup W$ is not subspace of V_2.

Note:- $U \cup W$ is not general a subspace. But we know that if S is nonempty subset of a vector space V, then [S] is the smallest subspace of V containing S.

$\therefore$ $[U \cup W]$ is the smallest subspace of V containing $U \cup W$.

Any element of $[U \cup W]$ is a linear combination of the finite subset of $U \cup W$.

i.e. if $v \in [U \cup W]$ then

$v = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n + \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_m v_m$

Here $u_1, u_2, \ldots, u_n \in U$, $v_1, v_2, \ldots, v_m \in W$and $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_m$ are scalars for n, m $\in$ N.

$\therefore$ v can be expressed as $u + v$ for $u \in$ U and $v \in$ W.

$\therefore$ We can say that $[U \cup W]$ consists of elements of the form u + v $u, \in$ U and $v \in$ W.

Definition:- Addition of sets:-

Let A and B be two subsets of a vector space V. As A + B, is the set of all vectors the for $u + v$, $u \in$ A and $v \in$ B.

i.e. A + B = { $u + v$/ $u \in$ A and $v \in$ B}

Example:- In V_2, let A = { (1, 2) ,(0, 1) } and B = { (1, 1) ,(-1, 2), (2, 5) }

then A + B = { (1, 2) + (1, 1) , (1, 2) +,(-1, 2), (1, 2) +(2, 5) ,

(0, 1) + (1, 1), (0, 1) +,(-1, 2), (0, 1) +(2, 5) }

= { (2, 3) , (0, 4), (3, 7), (1, 2), (-1, 3), (2, 6)}

Example:- In V_2, let A = { (2,3)} and B = { t(3, 1)/t a scalar } then

A + B = {(2,3) + t(3, 1) / t is scalar}

= {(2+3t, 3+ t) / t is scalar}

Geometric meaning of addition of sets:-

In V_2, let A = { (2,3)} and B = { t(3, 1)/t a scalar } then

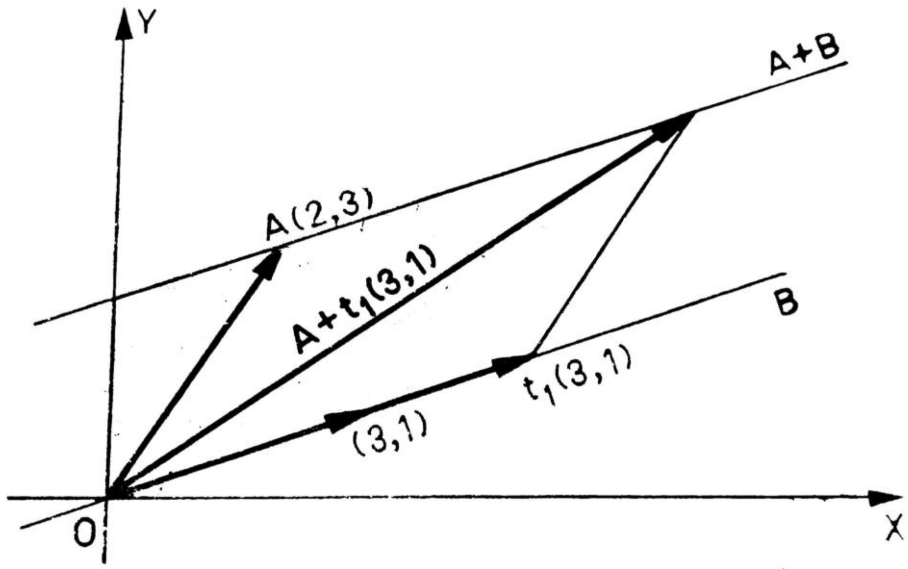
A + B = {(2,3) + t(3, 1) / t is scalar}

= {(2+3t, 3+ t) / t is scalar}

Geometric meaning of addition of sets A and B as under

B is a line through the origin and A is a set containing one vector.

A + B is line parallel to B and passing through the point (2,3)



Example:- In V_3, let A = { α(1, 2,0)/ α is a scalar} and

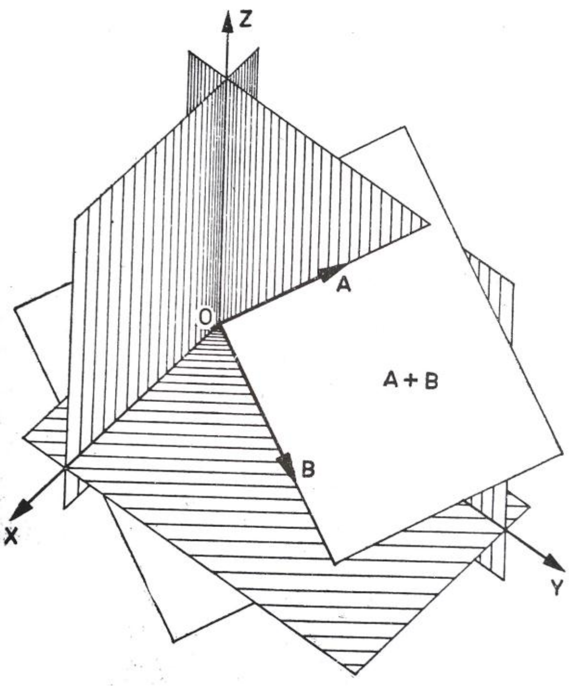
B ={ β(0, 1,2)/ β a scalar } then

A + B = {α(1, 2,0)+ β(0, 1,2) / α , β is scalar}

= {(α ,2α +β , 2β) / α , β is scalar}

Geometric meaning of addition of sets A and B as under

A and B are lines through the origin In V_3 and A + B is plane containing these lines and and passing through the point origin.



Theorem:- Let U and W are two subspaces of a vector space V then prove that U +W is subspace of V and U +W =[U $\cup$W].

Proof:- Since each vector of U +W is a finite linear combination of U$\cup$ W

U+ W$\subset$ [U$\cup$ W] -----------(i)

Now we want to prove that

[U$\cup$ W]$\subset$ U+ W

For this,

Let $v\in$ [U$\cup$ W]

$\therefore$ $v = u + w$ where $u\in$U and w$\in$W.

($\because$ Definition of addition of sets)

$\therefore$ $v\in$ U +W

$\therefore$ [U$\cup$ W]$\subset$ U+ W ------------(ii)

From (i) and (ii)

U +W =[U $\cup$W].

Since [U $\cup$W]. is subspace of V.

$\therefore$ U +W is subspace of V.

Note:- U +W is the smallest subspace of V containing U$\cup$ W. i.e. both U and W.

Example:- In V_3, U = x-axis and W = Y-axis, then find U +W.

Solution:- U +W is the set of all those vectors of V_3 that are from

α(1, 0,0)+ β(0, 1,0)

$\therefore$ U +W = {(α, β,0) / α, β are scalars}

And [U$\cup$ W] = {αu+βw / α, β is scalar and $u\in$U and $w\in$W}

i.e. [U$\cup$ W] = {α(1, 0,0)+ β(0, 1,0)/ α, β are scalars }

$\therefore$ U +W =[U$\cup$ W]

Note:- the interesting relation arising from this example:

[x-axis $\cup$ Y-axis] = x-axis + Y-axis = xy plane.

Direct sum

Definition:- Direct sum:-

Let U and W are subspaces of a vector space V, then the sum U +W is called direct sum if the sum U +W is subspace of V and U $\cap$W = v_0 ={0}.

It is denoted by U$\oplus$ W.

i.e. the direct sum of U and W is written as U$\oplus$ W.

Example:- Check the following additions in V_3.

(1) xy plane + z –axis = V_3.

(2) xy plane + yz plane = V_3.

Solution:- (1) xy plane + z –axis = V_3 is the direct sum
because xy plane $\cap$ z –axis ={0}.

(2) xy plane + yz plane = V_3. is not the direct sum
because xy plane $\cap$ yz plane $\neq$ {0}.

Note:- From example (1)Any vector (a, b, c) $\in V_3$ can be written as

(a, b, c) = (a, b, 0) + (0, 0, c) -----------(*)

Where (a, b, 0) $\in$ xy plane and (0, 0, c) $\in$ z axis

Thus (a, b, c) is the sum of two vectors one in the xy plane and the other is in z –axis .

$\therefore$ The advantage of the direct sum lies in the fact that the representation equation (*) is unique.

i.e. we cannot find two other vectors such that one in the xy- plane and the other is in z –axis .

On other hand, in example (2) any vector (a, b, c) can be written as the sum of two vectors, one in the xy-plane and other in the yz-plane in more than one way.

e.g. (a, b, c) = (a, b, 0) + (0, 0, c)

(a, b, c) = (a, 0, 0) + (0, b, c)

Theorem:- Let U and W are two subspaces of a vector space V and Z= U +W then Z= U $\oplus$ W iff the following condition is satisfied

Any vector $z \in$ Z can be expressed uniquely as the sum

$z = u + w$ for some $u \in$ U, $w \in$ W.

Proof:- Let Z= U $\oplus$ W

Since Z= U +W

If any vector $z \in$ Z can be written as $z = u + w$ for some $u \in$ U, $w \in$ W.

Let us suppose that $z = u' + w'$ for some $u' \in$ U, $w' \in$ W is another representation of z.

Then $u' + w' = u + w$

$\therefore u - u' = w' - w$

But U and W are subspaces of a vector space V.

$u - u' \in$ U and $w' - w \in$ W

$u - u' \in$ U $\cap$ W

Since U $\cap$ W is direct sum

$\therefore$ U $\cap$ W ={0}

$\therefore u - u' = 0 \Rightarrow u = u'$ and $w' = w$

$\therefore$ Any vector $z \in$ Z can be expressed uniquely as the sum

$z = u + w$ for some $u \in$ U, $w \in$ W.

Conversely,

Let us suppose that any vector $z \in$ Z can be expressed uniquely as the sum $z = u + w$ for some $u \in$ U, $w \in$ W.

Now we want to prove that Z is a direct sum of U and W.

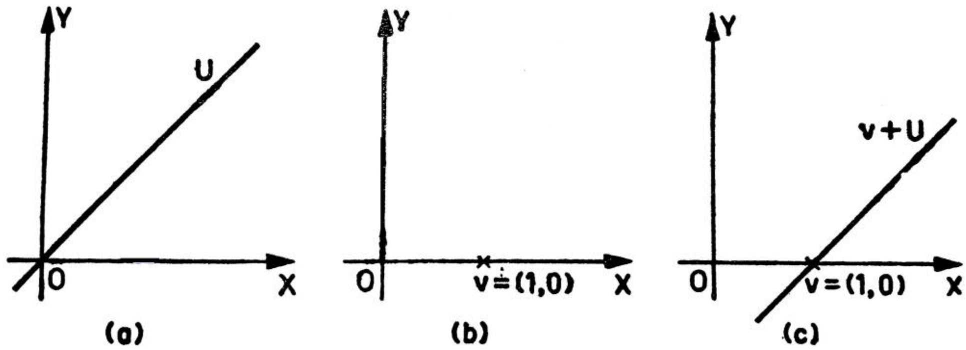
Since Z= U +W is given.
So we have only to prove that U $\cap$ W ={0}
Let us suppose that U $\cap$ W $\neq$ {0}
i.e. U $\cap$ W contain nonzero vector v.
i.e. $v \in$ U $\cap$ W where $v \neq 0$
$\therefore$ $v \in$ U and $v \in$ W
and $v = v + 0 \in$ U+ W with $v \in$ U, $0 \in$ W.
also $v = 0 + v \in$ U+ W with $0 \in$ U, $v \in$ W.
Thus, these two ways of expression of vector is not possible in direct sum.
$\therefore$ our supposition is wrong.
Hence U $\cap$ W ={0} and Z= U $\oplus$ W.

Definition:- Linear variety:-

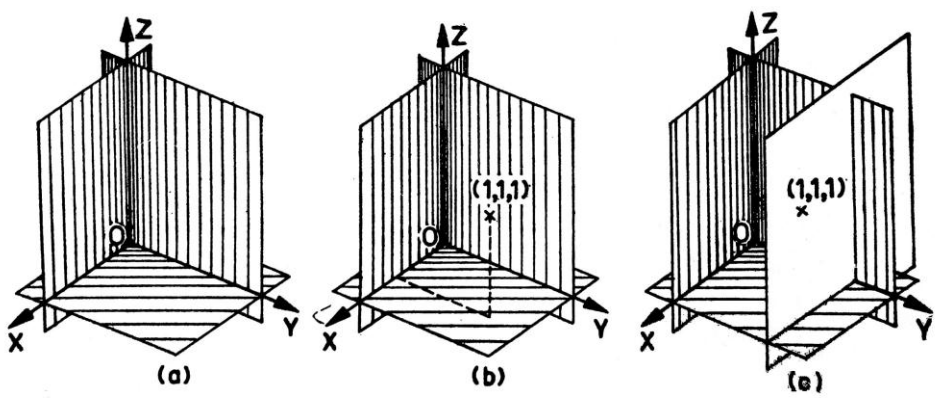
If U is a subspace of a vector space V and v a vector of V then {v} or v +U is called a **translate of U** (by v) or a **parallel of U** (through v) or linear variety.

Here U is called the base space of linear variety and v a leader.
{v}+ U is not a subspace unless $v \in$ U.

e.g. (1)Take the line $y = x$ through the origin in V_2.Call it U. Consider the point $v = (1, 0)$.
The translate v + U of U by v is the line $y = x-1$ through the point (1, 0) as in figure. It can also obtained by adding (1, 0) to the vectors in $y = x$.



e.g. (2)Let us take the plane $y = 0$ in V_3 and call it U.
Consider the point $v = (1, 1, 1) \in V_3$.
(1, 1, 1)+ U is the set of all points of V_3 given by (1, 1, 1)+ u $\forall u \in$ U



Geometrically

It is the plane parallel to y = 0 through the point (1, 1, 1).

Theorem:- Let U be a subspace of a vector space V and P = v +U be the parallel of U through v. Then prove that

(a) For any w in P, w + U = P. or Any vector of P can be taken as a leader of P.

(b) Two vectors $v_1, v_2 \in V$ are in the same parallel of U iff $v_1 - v_2 \in U$.

Proof:- (a) Let $w \in P$

Since P = v +U

$\therefore\ w = v + u_1$ where $u_1 \in U$.

$\therefore\ v = w - u_1$

Let us take $z \in P$ then $z = v + u_2$ where $u_2 \in U$.

$\therefore\ z = (w - u_1) + u_2$

$\quad = w + (u_2 - u_1)$

Here U is subspace of V.

$\therefore (u_2 - u_1) \in U$

Thus every vector $z \in P$ has the form w + (some vector in U).

$\therefore\ P \subset w + U$ ------ (1)

Now we want to prove that

$w + U \subset P$

For this,

Let $y \in w + U$

$\therefore\ y = w + u \quad \forall u \in U$

The vector

$y = w + u$

$\quad = v + u_1 + u$

$\quad = v$ + (a vector of U)

$\therefore\ y \in u + U = P$

$\therefore\ w + U \subset P$ ----------(2)

From (1) and (2)

$\therefore w + U = P$

Proof :- (b) Let v_1, v_2 be in the same parallel of U, namely v +U.

$\therefore\ v_1 = v + u_1$ where $u_1 \in U$ and $v_2 = v + u_2$ where $u_2 \in U$.

Then $v_1 - v_2 = (v + u_1) – (v + u_2) = u_1 – u_2$

Here U is subspace of V.

$\therefore\ u_1 – u_2 \in U$.

$\therefore\ v_1 - v_2 \in U$.

Conversely,

If $v_1 - v_2 \in U$ then $v_1 - v_2 = u$ for some $u \in U$.

So, $v_1 = v_2 + u$

$\therefore v_1 \in v_2 + U$

Also $v_2 = v_2 + 0$

$\therefore v_2 \in v_2 + U$ since $0 \in U$

So $v_1, v_2 \in V$ are in the same parallel of v_2 + U

Example:- Illustration:- Take V =V_3 and U = yz-plane.

Solution :- Let v = (1, 1,1) then p = v +U is the parallel given by the set

{ (1, 1,1) + (0, β,γ)/ β,γ arbitrary scalars}

= { (1, 1+β ,1+γ) / β,γ arbitrary scalars}

Part (a) of above theorem say that to describe this set we could take instead of (1, 1, 1) .any other vector from p.

Let us take vectoe (1, 0, 0). Which is also in p.

The theorem say that every vector (1, 1,1) + (0, β,γ) can be written in the form (1, 0,0) + (0, β',γ') for any β',γ'

β' =1+β and γ' = 1+γ

To continue the illustration, both (1, 1,1) and (1, 0,0) are in P

Part (b) of the theorem says that whenever the difference of two vectors belongs to U, then they both belong to the same parallel and conversely.