Matrices

Definition:- Matrix

Let U and V be vector spaces of dimensions n and m, respectively. Let $B_1 = \{ u_1, u_2, u_3, \ldots, u_n\}$ and $B_2 = \{ v_1, v_2, v_3, \ldots, v_m\}$ be ordered bases of U and V respectively. Let $T: U \rightarrow V$ be a linear map defined by $T(u_j) = \alpha_{1i} v_1 + \alpha_{2i} v_2 + \alpha_{3i}\ uv_3 + \ldots + \alpha_{mi} v_m$,j= 1, 2, 3, …,n so that the coordinate vector of $T(u_i)$ written as a column vector is $\begin{bmatrix} \alpha_{1j} \\ \alpha_{2j} \\ \alpha_{3j} \\ \vdots \\ \alpha_{mj} \end{bmatrix}$

Write the coordinate vectors of $T(u_1), T(u_2), \ldots, T(u_j), \ldots, T(u_n)$ successively as column vectors in the form of a rectangular array as follows:

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \ldots \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \ldots \alpha_{2n} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \ldots \alpha_{3n} \\ \vdots & \vdots & \vdots \ \vdots \ \vdots \\ \alpha_{m1} & \alpha_{m2} & \alpha_{m3} \ldots \alpha_{mn} \end{bmatrix}$$

This rectangular array is called the *matrix* of T relative to the *ordered base* B_1 and B_2, and is denoted by $(T: B_1, B_2) = (\alpha_{ij})_{m\times n}$.

Note:- in this definition α_{ij} is the *i*-th coordinate of $T(uj)$ relative to the basis $\{ v_1, v_2, v_3, \ldots, v_m\}$

→ The numbers that constitute a matrix are called is *entries*. Each horizontal line of entries is called a *row*. Each vertical line of entries is called a *column*.

Example:- Let a linear transformation $T: V_2 \rightarrow V_3$ be defined by $T(x_1, x_2) = (x_1+x_2, 2x_1-x_2, 7x_2)$. Find the matrix relative to the standard base.

Solution:- We know that the standard basis of v_2 is $B_1 = \{e_1, e_2\} = \{(1, 0), (0, 1)\}$ and the basis of v_3 is $B_2 = \{f_1, f_2, f_3\} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

$T(e_1) = T(1, 0) = (1+0, 2(1)-0, 7(0)) = (1, 2, 0)$

Now $(1, 2, 0) = \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3$

$= \alpha_1 (1,0,0) + \alpha_2 (0,1,0) + \alpha_3 (0,0,1)$

$= (\alpha_1, \alpha_2, \alpha_3)$

$\therefore\ \alpha_1 = 1,\ \alpha_2 = 2,\ \alpha_3 = 0$

$(1, 2, 0) = 1f_1 + 2f_2 + 0f_3$

$\therefore\ T(e_1) = 1f_1 + 2f_2 + 0f_3$ -------------(1)

$T(e_2) = T(0, 1) = (0+1, 2(0)-1, 7(1)) = (1, -1, 7)$

Now $(1, -1, 7) = \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3$

$= \alpha_1 (1,0,0) + \alpha_2 (0,1,0) + \alpha_3 (0,0,1)$

$= (\alpha_1, \alpha_2, \alpha_3)$

$\therefore\ \alpha_1 = 1,\ \alpha_2 = -1,\ \alpha_3 = 7$

$(1, -1, 7) = 1f_1 + (-1)f_2 + 7f_3$

$\therefore\ T(e_2) = 1f_1 + (-1)f_2 + 7f_3$ ------------------(2)

From equation (1) and (2)

The require matrix with respect to base B_1 and B_2 is

$$(T: B_1, B_2) = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 0 & 7 \end{bmatrix}_{3\times 2}$$

Example:- Let a linear transformation $T: V_3 \to V_3$ be defined by

$T(x_1, x_2, x_3) = (x_1 - x_2 + x_3, 2x_1 - 3x_2 - \frac{1}{2}x_3, x_1 - x_2 - 2x_3)$. Find the matrix relative to the

$B_1 = \{e_1, e_2, e_3\}$ and the basis of v_3 is $B_2 = \{(1, 1, 0), (1, 2, 3), (-1, 0, 1)\}$

Solution:- Here given basis are as

$B_1 = \{e_1, e_2, e_3\} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and the basis

$B_2 = \{f_1, f_2, f_3\} = \{(1, 1, 0), (1, 2, 3), (-1, 0, 1)\}$

Now $T(e_1) = T(1, 0,0) = (1 - 0 + 0, 2(1) - 0 - 0, 1 - 0 - 0) = (1, 2, 1)$

$T(e_1) = (1, 2, 1) = \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3$

$= \alpha_1 (1,1,0) + \alpha_2 (1,2,3) + \alpha_3 (-1,0,1)$

$= (\alpha_1 + \alpha_2 - \alpha_3, \alpha_1 + 2\alpha_2, 3\alpha_2 + \alpha_3)$

$\therefore \alpha_1 + \alpha_2 - \alpha_3 = 1, \alpha_1 + 2\alpha_2 = 2, 3\alpha_2 + \alpha_3 = 1$

Solving these equation we get

$\alpha_1 = 2, \alpha_2 = 0, \alpha_3 = 1$

$(1, 2, 1) = 2f_1 + 0f_2 + 1f_3$

$\therefore T(e_1) = T(1, 0,0) = 1f_1 + 2f_2 + 0f_3$ -------------(1)

Now $T(e_2) = T(0, 1,0) = (0 - 1 + 0, 2(0) - 3(1) - 0, 0 - 1 - 0) = (-1, -3, -1)$

$T(e_2) = (-1, -3, -1) = \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3$

$= \alpha_1 (1,1,0) + \alpha_2 (1,2,3) + \alpha_3 (-1,0,1)$

$= (\alpha_1 + \alpha_2 - \alpha_3, \alpha_1 + 2\alpha_2, 3\alpha_2 + \alpha_3)$

$\therefore \alpha_1 + \alpha_2 - \alpha_3 = -1, \alpha_1 + 2\alpha_2 = -3, 3\alpha_2 + \alpha_3 = -1$

Solving these equation we get

$\alpha_1 = 6, \alpha_2 = \frac{-3}{2}, \alpha_3 = \frac{11}{2}$

$(1, 2, 1) = 6f_1 + \frac{-3}{2} f_2 + \frac{11}{2} f_3$

$\therefore T(e_2) = 6f_1 + \frac{-3}{2} f_2 + \frac{11}{2} f_3$------------------(2)

Now $T(e_3) = T(0, 0,1) = (0 - 0 + 1, 2(0) - 3(0) - \frac{1}{2}, 0 - 0 - 2) = (1, -\frac{1}{2}, -2)$

$T(e_3) = (1, -\frac{1}{2}, -2) = \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3$

$= \alpha_1 (1,1,0) + \alpha_2 (1,2,3) + \alpha_3 (-1,0,1)$

$= (\alpha_1 + \alpha_2 - \alpha_3, \alpha_1 + 2\alpha_2, 3\alpha_2 + \alpha_3)$

$\therefore \alpha_1 + \alpha_2 - \alpha_3 = 1, \alpha_1 + 2\alpha_2 = -\frac{1}{2}, 3\alpha_2 + \alpha_3 = -2$

Solving these equation we get

$\alpha_1 = 0,\ \alpha_2 = \frac{-1}{4},\ \alpha_3 = \frac{-3}{4}$

$(1, -\frac{1}{2}, -2) = 0f_1 + \frac{-1}{4}f_2 + \frac{-3}{4}f_3$

$\therefore\ T(e_2) = 0f_1 + \frac{-1}{4}f_2 + \frac{-3}{4}f_3$------------------(3)

From equation (1) ,(2) and (3)

The require matrix with respect to base B_1 and B_2 is

$$(T: B_1, B_2) = \begin{bmatrix} 2 & 6 & 0 \\ 0 & -\frac{3}{2} & -\frac{1}{4} \\ 1 & \frac{11}{2} & -\frac{3}{4} \end{bmatrix}_{3\times 3}$$

Example:- Let a linear transformation $T: V_3 \rightarrow V_2$ be defined by

$T(e_1) = 2f_1 - f_2$,

$T(e_2) = f_1 + 2f_2$ and

$T(e_3) = 0f_1 + 0f_2$

Find the matrix relative to the standard base.

Solution:- Here $\{e_1, e_2, e_3\}$ and $\{f_1, f_2\}$ are standard base for V_3 and V_2 respectively.

$B_1 = \{e_1, e_2, e_3\} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and the basis

$B_2 = \{f_1, f_2\} = \{(1, 0), (0, 1)\}$

Let $(x_1, x_2, x_3) \in v_3$

Now $\mathbf{T}(x_1, x_2, x_3) = T(x_1e_1 + x_2e_2 + x_3e_3)$

$$\begin{aligned}\therefore T(x_1, x_2, x_3) &= x_1T(e_1) + x_2T(e_2) + x_3T(e_3) \\ &= x_1(2f_1 - f_2) + x_2(f_1 + 2f_2) + x_3(0f_1 + 0f_2) \\ &= (2x_1 + x_2)f_1 + (-x_1 + 2x_2)f_2 \\ &= (2x_1 + x_2)(1, 0) + (-x_1 + 2x_2)(0, 1) \\ &= (2x_1 + x_2, -x_1 + 2x_2)\end{aligned}$$

$\therefore T(x_1, x_2, x_3) = (2x_1 + x_2, -x_1 + 2x_2)$

Now $T(1, 0, 0) = (2, -1) = 2f_1 + (-1)f_2$

$\therefore\ T(e_1) = T(1, 0, 0) = 2f_1 + (-1)f_2$-------------(1)

also $T(0, 1, 0) = (1, 2) = 1f_1 + 2f_2$

$\therefore\ T(e_2) = T(0, 1, 0) = 1f_1 + 2f_2$-------------(2)

And $T(0, 0, 1) = (0, 0) = 0f_1 + 0f_2$

$\therefore\ T(e_3) = T(0, 0, 1) = 0f_1 + 0f_2$-------------(3)

From equation (1) ,(2) and (3)

The require matrix with respect to base B_1 and B_2 is

$$(T: B_1, B_2) = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 0 \end{bmatrix}_{2\times 3}$$

Example:- Let a linear transformation T: $V_3 \to V_2$ be defined by

$T(e_1) = 2f_1 - f_2$,

$T(e_2) = f_1 + 2f_2$ and

$T(e_3) = 0f_1 + 0f_2$

Find the matrix relative to basis $B_1 = \{(1,1,0), (1,0,1), (0,1,1)\}$ and the basis $B_2 = \{f_1, f_2, f_3\} = \{(1,1), (1,-1)\}$.

Solution: AS above example

$$(T: B_1, B_2) = \begin{bmatrix} 2 & \frac{1}{2} & \frac{3}{2} \\ -1 & \frac{3}{2} & -\frac{1}{2} \end{bmatrix}_{2\times 3}$$

Note: the matrix of T changes when we change the bases.

Example:- Let a linear transformation T: $\mathcal{P}_3 \to \mathcal{P}_2$ be defined by

$T(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3) = \alpha_3 + (\alpha_2 + \alpha_3)x + (\alpha_0 + \alpha_1)x^2$ .Let us calculate the matrix of T relative to the bases $B_1 = \{1, (x-1), (x-1)^2, (x-1)^3\}$ and $B_2 = \{1, x, x^2\}$.

Solution:- Now $T(1) = T(1 + 0x + 0x^2 + 0x^3) = 0 + (0+0)x + (1+0)x^2 = x^2$

$T(x-1) = T(-1 + 1x + 0x^2 + 0x^3) = 0 + (0+0)x + (-1+1)x^2 = 0$

$T((x-1)^2) = T(1 + (-2)x + 1x^2 + 0x^3) = 0 + (1+0)x + (1+-2)x^2. = x - x^2$

$T((x-1)^3) = T(-1 + 3x + (-3)x^2 + 1x^3) = 1 + (-3+1)x + (-1+3)x^2. = 1 - 2x + 2x^2$

Since $T(1) = x^2 = 0.1 + 0\,x + 1x^2$

$T(x-1) = 0 = 0.1 + 0\,x + 0.x^2$

$T((x-1)^2) = x - x^2 = 0.1 + 1\,x - 1.x^2$

$T((x-1)^3) = 1 - 2x + 2x^2 = 1.1 - 2\,x + 2.x^2$

Hence, the matrix of T relative to B_1 and B_2 is

$$(T: B_1, B_2) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 1 & 0 & -1 & 2 \end{bmatrix}_{3\times 4}$$

Example:- Let a linear transformation D: $\mathcal{P}_3 \to \mathcal{P}_2$ be defined by map D(p) = p′.

$T(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3) = \alpha_3 + (\alpha_2 + \alpha_3)x + (\alpha_0 + \alpha_1)x^2$ .Let us calculate the matrix of D relative to the bases $B_1 = \{1, x, x^2, x^3\}$ and $B_2 = \{1, x, x^2\}$.

Solution:- Since $D(1) = 0 = 0.1 + 0.\,x + 0.x^2$

$D(x) = 1 = 1.1 + 0\,x + 0.x^2$

$D(x^2) = 2\,x = 0.1 + 2\,x - 0.x^2$

$D(x^3) = 3x^2 = 0.1 + 0.\,x + 3.x^2$

Hence, the matrix of T relative to B_1 and B_2 is

$$(T: B_1, B_2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}_{3\times 4}$$

Note:- $\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}_{n\times n}$ is called identity matrix. and is denoted by *In* (or simply *I* , if n is understood).

The ij-th entry of this matrix is usually denoted by δ_{ij} ,where δ_{ij} ,called the *Kronecker delta*, is defined by

δ_{ij} = 1, if $i=j$

0, if $i \neq j$.

Linear map associated with a matrix:

Note:- Any rectangular array of numbers such as

$$\begin{bmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1j} & \cdots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2j} & \cdots & \beta_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_{i1} & \beta_{i2} & \cdots & \beta_{ij} & \cdots & \beta_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_{m1} & \beta_{m2} & \cdots & \beta_{mj} & \cdots & \beta_{mn} \end{bmatrix}_{m\times n}$$

is called m × n matrix. If m = n the matrix is called a square matrix.

* Two matrices A = $(\alpha_{ij})_{m\times n}$ and B = $(\beta_{ij})_{m\times n}$ are said to be equal if $\alpha_{ij} = \beta_{ij}$ for all *i* and *j* .
  It is written as A = B.
* A matrix whose entries are all real (complex) numbers is called a real (complex) matrix.

Now we will define a linear map from given matrix .

Example:- If matrix of a linear transformation T with respect to standard bases is

$\begin{bmatrix} 2 & 3 & -4 \\ 0 & 1 & 2 \end{bmatrix}_{2\times 3}$ then find image of (3, 1, -5) under T. Also define T.

i.e. find T(x, y, z)

Solution:- Given matrix is $\begin{bmatrix} 2 & 3 & -4 \\ 0 & 1 & 2 \end{bmatrix}_{2\times 3}$ and let {e_1, e_2, e_3} = {(1, 0, 0), (0, 1, 0), (0, 0, 1)} be a basis for V_3 .

$\therefore$ T: $V_3 \to V_2$ is linear map and

T(1, 0, 0) = (2, 0)

T(0, 1, 0) = (3, 1)

T(0, 0, 1) = (-4, 2)

$\therefore$ **T(**3, 1,-5)= T(3e_1+ 1 e_2 -5 e_3)

= 3T(e_1) + 1 T(e_2) - 5 T(e_3)

= 3(2, 0)+ 1 (3, 1) - 5 (-4, 2)

= (29, -9)

Now T(x, y, z) = T(x e_1+ y e_2 +z e_3)

= x T(e_1)+ y T(e_2)+z T(e_3)

$= x\,(2, 0)+ y\,(3, 1) +z\,(-4, 2) = (2x + 3y -4z, y +2z)$

$\therefore\ T(x, y, z) = (2x + 3y -4z, y +2z)$

Example:- find T(-2, 1, 0, 5) if matrix of a linear transformation T with respect to standard bases is

$$\begin{bmatrix} 5 & 2 & 4 & 1 \\ 0 & -1 & 2 & 3 \\ 1 & 2 & -1 & 0 \end{bmatrix}_{3\times 4}$$ Also define T. i.e. find $T(x, y, z)$

Solution:- as above T(-2, 1, 0, 5) = (-3, 14, 0)

Example:- If matrix of T with respect to bases B_1 and B_2 is $\begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}_{2\times 3}$

Where B_1 = {(1,2, 0), (0,-1, 0), (1,-1, 1)} and B_1 = {(1, 0), (2,-1)} then define T i.e. find $T(x, y, z)$.

Solution:-Given matrix is $\begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}_{2\times 3}$

let $B_1 = \{u_1, u_2, u_3\}$ ={(1,2, 0), (0,-1, 0), (1,-1, 1)} and $B_1 = \{f_1, f_2\}$ ={(1, 0), (2,-1)}

$\therefore T: V_3 \to V_2$ is linear map and

Now $T(u_1)= -1\,(f_1) + 1\,(f_2)$

$\therefore T(1,2, 0)= -(1, 0) + 1(2,-1)= (1, -1)$

$T(u_2)= 2(f_1) + 0(f_2)$

$\therefore T(0, -1, 0) = 2(1, 0) + 0(2,-1)= (2, 0)$

$T(u_2)= 1(f_1) + 3(f_2)$

$\therefore T(1, -1, 1) = 1(1, 0) + 3(2,-1)= (7, -3)$

Let $(x, y, z) = \alpha_1 (1,2, 0) + \alpha_2\ (0,-1, 0) + \alpha_3\ (1,-1, 1)$

$\therefore\ (\alpha_1 + \alpha_3\ ,2\alpha_1 +-\alpha_2 - \alpha_3\ ,\ \alpha_3\ )= (x, y, z)$

$\therefore\ \alpha_1 + \alpha_3 = x,\ \ 2\alpha_1 +-\alpha_2 - \alpha_3 = y\ ,\ \alpha_3 = z$

Solving this equations then we get

$\alpha_1 = x- z,\ \ \alpha_2 = 2x - y -3z,\ \alpha_3 = z$

$\therefore\ T(x, y, z) = \alpha_1 T(1,2, 0) + \alpha_2\ T(0,-1, 0) + \alpha_3 T\,(1,-1, 1)$

$= (x- z)\,(1, -1)+ (2x - y -3z)\,(2, 0)+ z\,(7, -3)$

$T(x, y, z) = (5x - 2y, -x - 2z)$

Example:- If matrix of T with respect to standard bases B_1 and B_2 is $\begin{bmatrix} 2 & -3 & 4 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \\ 1 & 2 & -2 \end{bmatrix}_{4\times 3}$

then define T i.e. find $T(x, y, z)$.

Solution:-Given matrix is $\begin{bmatrix} 2 & -3 & 4 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \\ 1 & 2 & -2 \end{bmatrix}_{4\times 3}$

let $B_1 = \{u_1, u_2, u_3\}$ and $B_2 = \{f_1, f_2, f_3, f_4\}$ standard bases V_3 and V_4 of respectively.

$\therefore$ T: $V_3 \to V_4$ is linear map and

Now $T(u_1) = 2(f_1) + 1(f_2) + (-2)(f_3) + 1(f_4)$

$\therefore T(1,0,0) = 2(1,0,0,0) + 1(0,1,0,0) + (-2)(0,0,1,0) + 1(0,0,0,1)$

$= (2, 1, -2, 1)$

$\therefore T(u_1) = (2, 1, -2, 1)$

Now $T(u_2) = -3(f_1) + 0(f_2) + 1(f_3) + 2(f_4)$

$\therefore T(0,1,0) = -3(1,0,0,0) + 0(0,1,0,0) + 1(0,0,1,0) + 2(0,0,0,1)$

$= (-3, 0, 1, 2)$

$\therefore T(u_2) = (-3, 0, 1, 2)$

Now $T(u_3) = 4(f_1) + -1(f_2) + 0(f_3) + -2(f_4)$

$\therefore T(0,1,0) = 4(1,0,0,0) + -1(0,1,0,0) + 0(0,0,1,0) + -2(0,0,0,1)$

$= (4, -1, 0, -2)$

$\therefore T(u_3) = (4, -1, 0, -2)$

Let $(x, y, z) = \alpha_1(1,0,0) + \alpha_2(0,1,0) + \alpha_3(0,0,1)$

$\therefore (\alpha_1, \alpha_2, \alpha_3) = (x, y, z)$

$\therefore \alpha_1 = x, \quad \alpha_2 = y, \ \alpha_3 = z$

$\therefore T(x, y, z) = \alpha_1 T(1,0,0) + \alpha_2 T(0,1,0) + \alpha_3 T(0,0,1)$

i.e $T(x, y, z) = \alpha_1 T(u_1) + \alpha_2 T(u_2) + \alpha_3 T(u_3)$

$= (x)(2, 1, -2, 1) + (y)(-3, 0, 1, 2) + z(4, -1, 0, -2)$

$T(x, y, z) = (2x - 3y + 4z, x - z, -2x + y + 2z, x + 2y - 2z)$

Note:- let $\mathcal{M}_{m,n}$ denote the set of all $m \times n$ real matrices. Let U and V be real vector spaces of dimensions n and m, respectively. Let fix ordered bases B_1 for U and B_2 for V. Then the process of determining the matrix of a linear map and the linear map corresponding to a matrix show that the map $\tau : L(U,V) \to \mathcal{M}_{m,n}$ defined by $\tau(T) = (T: B_1, B_2)$ is one-one and onto.

- Here $\tau(T) = (T:, B_1, B_2)$ is one-one and onto therefore τ is isomorphism.
- Here L(U, V) is the set of all linear transformations from U to V.
- Since each linear map T: U$\to$V there exists a unique linear map from U to V.
- When $U = U_n$ and $V = V_m$ and the bases B_1 and B_2 are standard bases in the spaces, then the matrix associated with $T: U_n \to V_m$ is called its natural matrix.
- Here $\tau(0) = 0_{m \times n}$
- If $\tau : L(U) \to \mathcal{M}_{n,n}$. then $\tau(I) = I_n$ where I is the identity transformation on U and I_n is the $n \times n$ identity matrix.

Linear operations in $\mathcal{M}_{m,n}$

Definition:- Sum of two matrices (of the same order):

Let A = $(\alpha_{ij})_{m\times n}$ and B = $(\beta_{ij})_{m\times n}$ be two matrices. Then the sum A + B is defined as the m × n matrix $(\alpha_{ij}+\beta_{ij})$

i.e. the sum of two matrices (of the same order) is obtained by adding the corresponding entries.

e.g. Let A = $\begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & 1 \end{bmatrix}$, B = $\begin{bmatrix} -1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ then A + B = $\begin{bmatrix} 1-1 & 3+2 & 2+1 \\ 0+3 & -1+1 & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 5 & 3 \\ 3 & 0 & 3 \end{bmatrix}$

Definition:- Scalar multiplication of matrix:

Let A = $(\alpha_{ij})_{m\times n}$ matrix and α be scalar. Then the Scalar multiplication of matrix α A is defined as the m × n matrix $(\alpha\ \alpha_{ij})$

i.e. The Scalar multiplication of matrix by α is obtained by multiplying each entries by α .

e.g. Let A = $\begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & 1 \end{bmatrix}$ and = 5 then α A =5 $\begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 15 & 10 \\ 0 & -5 & 5 \end{bmatrix}$

Example:- prove that the linear map $\tau : L(U,V) \to \mathcal{M}_{m,n}$ defined by τ (T) =(T:B_1, B_2) is linear.

Solution:- Let fix ordered bases B_1 for U and B_2 for V. Where $B_1 = \{u_1, u_2, \ldots, u_n\}$ $B_2 = \{v_1, v_2, \ldots, v_n\}$.

Let T, S $\in$ L(U, V)

i.e. T: U $\to$ V and S: U $\to$ V be linear map. Then T + S is also linear map.

If matrices of T and S relative to B_1 and B_2 are A = $(\alpha_{ij})_{m\times n}$ and B = $(\beta_{ij})_{m\times n}$ respectively. Then $T(u_j) = \sum_{i=1}^{m} \alpha_{ij} v_i$ (j = 1, 2, 3, …, n)

And $S(u_j) = \sum_{i=1}^{m} \beta_{ij} v_i$ (j = 1, 2, 3, …, n)

Here $T(u_j) + S(u_j) = \sum_{i=1}^{m} \alpha_{ij} v_i + \sum_{i=1}^{m} \beta_{ij} v_i$ (j = 1, 2, 3, …, n)

$= \sum_{i=1}^{m} (\alpha_{ij}+\beta_{ij}) v_i$ (j = 1, 2, 3, …, n) --------(1)

$(T + S)(u_j) = \sum_{i=1}^{m} (\alpha_{ij} v_i + \beta_{ij} v_i)$ (j = 1, 2, 3, …, n)

$= \sum_{i=1}^{m} (\alpha_{ij}+\beta_{ij}) v_i$ (j = 1, 2, 3, …, n) --------(2)

From (1) and (2) we get

$(T + S)(u_j) = T(u_j) + S(u_j)$ (j = 1, 2, 3, …, n)

$\therefore \tau$ (T +S) = τ (T) + τ (S) (j = 1, 2, 3, …, n) --------(3)

Also $T(\alpha u_j) = \sum_{i=1}^{m} \alpha\alpha_{ij} v_i = \alpha \sum_{i=1}^{m} \alpha_{ij} v_i = \alpha\, T(u_j) = (j = 1, 2, 3, \ldots, n)$

$\therefore (\alpha\ \tau)(T) = \tau(\alpha T)$ for scalar α .--------(4)

From (3) and (4) we get

the linear map $\tau : L(U,V) \to \mathcal{M}_{m,n}$ defined by $\tau(T) = (T:,B_1, B_2)$ is linear.

Note: the linear map $\tau : L(U,V) \to \mathcal{M}_{m,n}$ preserves addition and scalar multiplication.

i.e. $\tau(T + S) = \tau(T) + \tau(S)$ for all T, S $\in$ L(U, V) and $(\alpha\ \tau)(T) = \tau(\alpha T)$ for scalar α .

Theorem: (a)Prove that is a real vector space for the foregoing definitions of addition and scalar multiplication.

(b) prove $\tau : L(U,V) \to \mathcal{M}_{m,n}$ defined by $\tau(T) = (T:B_1, B_2)$ is isomorphism.

Theorem: (Dimension theorem for $\mathcal{M}_{m,n}$)

The dimension on of the vector space $\mathcal{M}_{m,n}$ is mn.

Proof:- Given i and j, define the matrix E_{ij} as the m × n matrix with 1 in the ij-th entry and zero in all other entries.

Let set B of matrices as

$B = \{ E_{11},\ldots,E_{1n}, E_{21},\ldots,E_{2n},\ldots, E_{m1},\ldots,E_{mn},\}$

$= \{ E_{ij} \in \mathcal{M}_{m,n} \,/\, i = 1, 2, 3, \ldots, m ;\, j = 1, 2, 3, \ldots, n\}$

Now we want to prove that B is a basis for $\mathcal{M}_{m,n}$

First we will show that B is LI.

We assume that

$\alpha_{11}E_{11}+\ldots+ \alpha_{1n}E_{1n}+\alpha_{21}E_{21}+\ldots +\alpha_{2n}E_{2n}+\ldots+\alpha_{m1}E_{m1}+\ldots+\alpha_{mn}E_{mn}= 0_{m\times n}$

i.e.

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1j} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2j} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{i1} & \alpha_{i2} & \cdots & \alpha_{ij} & \cdots & \alpha_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mj} & \cdots & \alpha_{mn} \end{bmatrix}_{m\times n} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}_{m\times n}$$

Hence $\alpha_{ij} = 0$ for all i and j