Sem-III
MAT 202: Linear Algebra-I
Unit-3 Linear transformations

Definition:- Linear transformation :-

Suppose U and V are vector spaces either both real or both complex. Then the map T: U$\to$ V is said to be a linear map (transformation, operator), if

(i) T (u_1 + u_2) = T (u_1) + T (u_2) for all, u_1 , $u_2 \in$ U

(ii) T (α u) = α T (u) for all, u $\in$ U and all scalars α .

Note:- A linear map T: U$\to$ U is said to be a linear map on U.

Whenever we say T: U$\to$ U is a linear map, then U and V shall be taken as vector spaces over the same field of scalars.

Example:- Prove that the map T: $V_3 \to V_3$ define by T $(x_1,x_2,x_3) = (x_1,x_2,0)$ linear map.

Solution:- Let α be any scalar and $x, y \in V_3$ where $x=(x_1, x_2, x_3)$ and $y = (y_1,y_2,y_3)$

$\therefore x + y =(x_1, x_2, x_3) + (y_1,y_2,y_3)= (x_1+ y_1, x_2+ y_2 , x_3+ y_3)$

And $\alpha\ x=(\alpha x_1, \alpha\ x_2, \alpha\ x_3)$

Now $T(x + y) = T(x_1+ y_1, x_2+ y_2 , x_3+ y_3)$

$= (x_1+ y_1, x_2+ y_2, 0)$ ____________(1)

($\because$ by definition of T)

Now T(x) + T(y) = $T(x_1, x_2, x_3) + T(y_1,y_2,y_3)$

$= (x_1, x_2, 0) + (y_1,y_2,0)$ ($\because$ by definition of T)

$= (x_1+ y_1, x_2+ y_2, 0)$ ____________(2)

$T(\alpha x)= T(\alpha x_1, \alpha\ x_2, \alpha\ x_3) = (\alpha x_1, \alpha\ x_2, 0)$

$= \alpha (x_1,x_2,0)$ _________(3)

$\alpha T(x)=\alpha T (x_1,x_2,x_3) = \alpha (x_1,x_2,0)$____________(4)

From (1) ,(2),(3) and (4)

T: $V_3 \to V_3$ linear map.

Note:- T: $V_3 \to V_3$ define by T $(x_1, x_2, x_3) = (x_1, x_2, 0)$ is called the projection on x_1x_2 plane.

Example:- Prove that the map T: $V_3 \to V_2$ define by T $(x_1,x_2,x_3) = (x_1 - x_2, x_1+x_3)$ linear map.

Solution:- Let α be any scalar and $x, y \in V_3$ where $x=(x_1, x_2, x_3)$ and $y = (y_1,y_2,y_3)$

$\therefore x + y =(x_1, x_2, x_3) + (y_1, y_2, y_3)= (x_1+ y_1, x_2+ y_2 , x_3+ y_3)$

And $\alpha\ x=(\alpha x_1, \alpha\ x_2, \alpha\ x_3)$

Now $T(x + y) = T(x_1+ y_1, x_2+ y_2 , x_3+ y_3)$

$= (x_1+ y_1 - x_2- y_2, x_1+ y_1+ x_3+ y_3)$ ____________(1)

($\because$ by definition of T)

Now T(x) + T(y) = $T(x_1, x_2, x_3) + T(y_1,y_2,y_3)$

$= (x_1 - x_2, x_1+x_3) + \quad (y_1-y_2, y_1+y_3)$ ($\because$ by definition of T)

$= (x_1+ y_1 - x_2- y_2, x_1+ y_1+ x_3+ y_3)$ ____________(2)

$T(\alpha x)= T(\alpha x_1, \alpha x_2, \alpha x_3) = (\alpha x_1-\alpha x_2, \alpha x_1+\alpha x_3)$

$= \alpha (x_1 - x_2, x_1+x_3)$ _________(3)

$\alpha T(x)=\alpha T (x_1,x_2,x_3) = \alpha (x_1 - x_2, x_1+x_3)$ ____________(4)

From (1) ,(2),(3) and (4)

T: $V_3 \rightarrow V_2$ linear map.

Example:- Examine the map T: $V_3 \rightarrow V_1$ define by T $(x_1,x_2,x_3) = (x_1^2 + x_2^2 + x_3^2)$ linear map or not.

Solution:- Let α be any scalar and $x, y \in V_3$ where $x=(x_1, x_2, x_3)$ and $y = (y_1,y_2,y_3)$

$\therefore x + y =(x_1, x_2, x_3) + (y_1, y_2, y_3)= (x_1+ y_1, x_2+ y_2 , x_3+ y_3)$

And $\alpha x=(\alpha x_1, \alpha x_2, \alpha x_3)$

Now $T(x + y) = T(x_1+ y_1, x_2+ y_2 , x_3+ y_3)$

$= ((x_1+ y_1)^2 + (x_2+ y_2)^2 + (x_3+ y_3)^2)$ ____________(1)

($\because$ by definition of T)

Now $T(x) + T(y) = T(x_1, x_2, x_3) + T(y_1,y_2,y_3)$

$= (x_1^2 + x_2^2 + x_3^2) + (y_1^2 + y_2^2 + y_3^2)$ ($\because$ by definition of T)

____________(2)

From (1) and (2)

$\therefore$ $T(x + y) \neq T(x) + T(y)$

T: $V_3 \rightarrow V_1$ is not linear map.

Example:- Prove that the map T: $U \rightarrow V$ define by T $(u) = 0_v$ linear map.

Solution:- Let α be any scalar and $x, y \in U$

Now $T(x + y) = 0_v$ _________(1)

($\because$ by definition of T)

Now $T(x) + T(y) = 0_v + 0_v = 0_v$ ($\because$ by definition of T)

____________(2)

$T(\alpha x)= 0_v$ _________(3)

$\alpha T(x)=\alpha\ 0_v = 0_v$ ____________(4)

From (1) ,(2),(3) and (4)

T: $V_3 \rightarrow V_2$ linear map.

Note:- the map T: $U \rightarrow V$ define by T $(u) = 0_v$ is called zero map.

Example:- Prove that the map T: $U \rightarrow U$ define by T $(u) = u$ linear map.

Solution:- Let α be any scalar and $x, y \in U$

Now $T(x + y) = x + y$ _________(1)

($\because$ by definition of T)

Now $T(x) + T(y) = x + y$ ($\because$ by definition of T)

____________(2)

$T(\alpha x)= \alpha x$ _________(3)

$\alpha T(x)=\alpha x= \alpha x$ ____________(4)

From (1) ,(2),(3) and (4)

T: $U \rightarrow U$ linear map.

Note:- the map T: U $\rightarrow$ U define by T (u) = u is called identity map.

Example:- Prove that the map T: $V_2 \rightarrow V_2$ define by T $(x_1,x_2) = (x_1, -x_2)$ linear map.

Solution:- Let α be any scalar and $x, y \in V_2$ where $x=(x_1, x_2)$ and $y = (y_1,y_2)$

$\therefore x + y = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$

And $\alpha x = (\alpha x_1, \alpha x_2)$

Now $T(x + y) = T(x_1 + y_1, x_2 + y_2)$

$= (x_1 + y_1, - x_2 - y_2)$ ____________(1)

($\because$ by definition of T)

Now $T(x) + T(y) = T(x_1, x_2) + T(y_1,y_2)$

$= (x_1 - x_2) + (y_1 - y_2)$ ($\because$ by definition of T)

$= (x_1 + y_1, - x_2 - y_2)$ ____________(2)

$T(\alpha x) = T(\alpha x_1, \alpha x_2) = (\alpha x_1, -\alpha x_2)$

$= \alpha (x_1, - x_2)$ _________(3)

$\alpha T(x) = \alpha T(x_1,x_2) = \alpha (x_1, - x_2)$ ____________(4)

From (1) ,(2),(3) and (4)

T: $V_2 \rightarrow V_2$ linear map.

Note:- the map T: $V_2 \rightarrow V_2$ define by T $(x_1,x_2) = (x_1, - x_2)$ is called the reflection in the x_1-axis.

Figure

Example:- Prove that the map D: $\mathcal{C}^{(1)}(a,b) \rightarrow \mathcal{C}^{(1)}(a,b)$ define by D (f) = $f\,'$ linear map. Where is the derivative of f.

Solution:- Let α be any scalar and $f, g \in \mathcal{C}^{(1)}(a,b)$

Now $T(f + g) = (f + g)\,'$

$= f\,' + g\,'$____________(1)

($\because$ by definition of T)

Now $T(f) + T(g) = f\,' + g\,'$ ____________(2)

($\because$ by definition of T)

$T(\alpha f) = (\alpha f)' = \alpha (f)'$ _________(3)

$\alpha T(f) = \alpha (f)'$____________(4)

From (1) ,(2),(3) and (4)

D: $\mathcal{C}^{(1)}(a,b) \rightarrow \mathcal{C}^{(1)}(a,b)$ linear map.

Example:- Prove that the map D: $\mathcal{C}^{(1)}(a,b) \rightarrow$ R define by $\mathcal{J}(f) = \int_a^b f(x)dx$ linear map.

Solution:- Let α be any scalar and $f, g \in \mathcal{C}^{(1)}(a,b)$

Now $T(f + g) = \int_a^b (f(x) + g(x))dx$

$= \int_a^b f(x)dx + \int_a^b g(x)dx$ ____________(1)

($\because$ by definition of T)

Now $T(f) + T(g) = \int_a^b f(x)dx + \int_a^b g(x)dx$ ____________(2)

($\because$ by definition of T)

$T(\alpha f) = \int_a^b \alpha f(x)dx = \alpha \int_a^b f(x)dx$ _________(3)

$\alpha T(f) = \alpha \int_a^b f(x)dx$ ____________(4)

From (1) ,(2),(3) and (4)

D: $\mathcal{C}^{(1)}(a,b) \rightarrow R$ linear map.

Example:- Prove that the map T: U$\rightarrow$ U define by $T(x) = x + u_0$ is not linear map. Where u_0 is a fixed vector in U.

Solution:- Let α be any scalar and $x, y \in U$

Now $T(x + y) = x + y + u_0$ _________(1)

($\because$ by definition of T)

Now $T(x) + T(y) = x + u_0 + y + u_0$ ($\because$ by definition of T) ____________(2)

From (1) and (2)

$\therefore T(x + y) \neq T(x) + T(y)$

T: U$\rightarrow$ U is not linear map.

Note:- the map T: U$\rightarrow$ U define by $T(x) = x + u_0$ is called translation by the vector u_0 .Where u_0 is a fixed vector in U.

- The function f: R$\rightarrow$ R defined by $f(x) = x + a$ ('a' fixed) is called a linear function, because its graph in xy-plane is straight line. But it is not a linear map from the vector space V_1 to itself.

Theorem:- let T: U$\rightarrow$ V be a linear map, then

(a) $T(0_u) = 0_V$

(b) $T(-u) = -T(u)$

(c) $T(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \ldots + \alpha_n u_n)$
$= \alpha_1 T(u_1) + \alpha_2 T(u_2) + \alpha_3 T(u_3) + \ldots + \alpha_n T(u_n)$.

i.e. A linear map T transforms the zero vector of U into the zero vector of V and negative of every u of U into the negative of T(u) of V.

Proof:- (a) $T(0_u) = T(0.u)$ [$\because$ $0.u = 0, u \in U$]

$= 0T(u)$ [$\because$ T is linear]

$= 0_V$

(b) $T(-u) = T(-1.u)$ [$\because$ $(-1).u = -u, u \in U$]

$= (-1)T(u)$ [$\because$ T is linear]

$= -T(u)$

(c) This result can be proved by mathematical induction.

Let p(n): $T(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \ldots + \alpha_n u_n)$

$= \alpha_1 T(u_1) + \alpha_2 T(u_2) + \alpha_3 T(u_3) + \ldots + \alpha_n T(u_n)$.

Then p(1): $T(\alpha_1 u_1) = \alpha_1 T(u_1)$

Since T is linear this is obviously true.

So the result is true for n =1.

Assume that p(k) to be true

i.e. p(k): $T(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \ldots + \alpha_n u_k)$

$= \alpha_1 T(u_1) + \alpha_2 T(u_2) + \alpha_3 T(u_3) + \ldots + \alpha_n T(u_k)$. is true.

We try to establish the result for n =k + 1

$T(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \ldots + \alpha_{k+1} u_{k+1})$

$= \alpha_1 T(u_1) + \alpha_2 T(u_2) + \alpha_3 T(u_3) + \ldots + \alpha_{k+1} T(u_{k+1})$.

By the hypothesis and linearity of T.

Since (i) p(1) is true.

(ii) p(k) $\Rightarrow$ p(k+1)

The result is true for all n.

Theorem:- A linear transformation T is completely determined by the values of elements of a basis. Precisely, if $B=\{u_1,u_2,\ldots,u_n\}$ is a basis for U and $v_1,v_2,\ldots,v_n$ be n vectors (not necessarily distinct) in V, then there exists a unique linear transformation $T:U \to V$ such that $T(u_i) = v_i$ for i=1,2,…,n.

Proof:- Let $u \in U$. since $B=\{ u_1,u_2,\ldots,u_n\}$ is a basis for U ,any vector u in U can be written as a unique linear combination of basis elements. Hence there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ satisfying

$u = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n$.

We define mapping $T: U \to V$ by $T(u) = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$.

We prove the following facts:

(i) T is linear transformation.

(ii) $T(u_i) = v_i$

(iii) Such mapping T is unique.

Proof of (i):- Let $u,v \in U$. Then there are scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_n$ for which

$u = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n$.

$v = \beta_1 u_1 + \beta_2 u_2 + \ldots + \beta_n u_n$.

$u + v = (\alpha_1 + \beta_1) u_1 + (\alpha_2 + \beta_2) u_2 + \ldots + (\alpha_n + \beta_n) u_n$

By definition of T

$T(u) = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$.

T(v)= β_1 v$_1$+β_2 v$_2$+…+β_n v$_n$.

T(u+v)= $(\alpha_1+\beta_1)$ v$_1$+$(\alpha_2+\beta_2)$ v$_2$+…+$(\alpha_n+\beta_n)$ v$_n$

It is clear that

T(u+v)= T(u) +T(v)

Also we can easily show that

T(α u)= α T(u)

For every scalar α and every vector u$\in$U. This establishes that T is a linear transformation from U to V.

(ii) Now $u_i \in B$, i=1,2,…,n

So, u_i can be expressed in terms B as $u_i = 0.u_1 + 0.u_2 + 0.u_3 + \ldots + 0.u_n$

$T(u_i) = 0.v_1 + 0.v_2 + 0.v_3 + \ldots + 0.v_n$

$= v_i$, i= 1, 2, 3, …, n.

(iii) Let S: U$\rightarrow$V be any other linear transformation define by

$S(u_i) = v_i$ i= 1, 2, 3, …, n.

Now S(u) = S($\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n$)

$= \alpha_1 S(u_1) + \alpha_2 S(u_2) + \alpha_3 S(u_3) + \ldots + \alpha_n S(u_n)$.

$= \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$

= T(u).

$\therefore$ S(u) = T(u).

This proved that such mapping T is unique.

Example:- If T is a linear transformation from $V_2 \rightarrow V_2$ define by

T (2,1) = (3, 4)

T (-3,4) = (0, 5)

then express (0, 1) as a linear combination of (2, 1) and (-3, 4). Hence find image of (0, 1) under T.

Solution:- Let (0, 1) =α (2, 1) +β (-3, 4)

$\therefore (2\alpha - 3\beta,\ \alpha + 4\beta) = (0, 1)$

$\therefore 2\alpha - 3\beta = 0,\ \alpha + 4\beta = 1$

Solving these equation then we get $\alpha = \frac{3}{11}$, $\beta = \frac{2}{11}$

$\therefore (0, 1) = \frac{3}{11}(2, 1) + \frac{2}{11}(-3, 4)$.

$$\therefore T(0, 1) = T\left(\frac{3}{11}(2, 1) + \frac{2}{11}(-3, 4)\right)$$

$$= \frac{3}{11} T(2, 1) + \frac{2}{11} T(-3, 4)$$

$$= \frac{3}{11}(3, 4) + \frac{2}{11}(0, 5) = \frac{1}{11}(9, 22)$$

Thus we get $T(0, 1) = \frac{1}{11}(9, 22)$

Example:- If T is a linear transformation from $R^3 \rightarrow R^3$ define by

$T(e_1) = e_1 + e_2 + e_3$,$T(e_2) = e_2 + e_3$ and $T(e_3) = e_2 - e_3$ where e_1, e_2, e_3 are unit vector of R^3 . Then (i)Determine the transformation of (2, -1, 3) And (ii)describe explicitly the linear transformation T.

Solution:- Since e_1, e_2, e_3 are unit vector of R^3

$\therefore\ e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$

We have $T(e_1) = e_1 + e_2 + e_3 \Rightarrow T(1, 0, 0) = (1, 0, 0) + (0, 1, 0) + (0, 0, 1)$

$= (1, 1, 1)$

$T(e_2) = e_2 + e_3 \Rightarrow T(0, 1, 0) = (0, 1, 0) + (0, 0, 1)$

$= (0, 1, 1)$

$T(e_3) = e_2 - e_3 \Rightarrow T(0, 0, 1) = (0, 1, 0) - (0, 0, 1)$

$= (0, 1, -1)$

Since e_1, e_2, e_3 form basis for R^3 .

$\therefore$ every vector of R^3can be uniquely expressed as a linear combination of e_1, e_2, e_3.

(i) Now $(2, -1, 3) = 2(1, 0, 0) + (-1)(0, 1, 0) + 3(0, 0, 1) = 2e_1 + (-1)e_2 + 3e_3$

$\therefore T(2, -1, 3) = 2T(e_1) + (-1)T(e_2) + 3T(e_3)$

$= 2(1, 1, 1) + (-1)(0, 1, 1) + 3(0, 1, -1)$

$= (2, 4, -2)$

(ii) $(x, y, z) \in R^3$.

Now $(x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1) = xe_1 + ye_2 + ze_3$

$\therefore T(x, y, z) = xT(e_1) + yT(e_2) + zT(e_3)$

$= x(1, 1, 1) + y(0, 1, 1) + z(0, 1, -1)$

$= (x, x + y + z, x + y - z)$

$\therefore T(x, y, z) = (x, x + y + z, x + y - z)$

Which is require linear transformation T.

Range and Kernel of a Linear map

Definition:-Kernel of a Linear map (null space)

Let $T: U \rightarrow V$ be a linear map. The Kernel (null space) of T is the set

$N(T) = \{u \in U / T(u) = 0\}$.

It is denoted as kerT.

OR N(T) is the set of all those elements in U that are mapped by T into the zero of V.

Definition:- Range of a Linear map (null space)

Let $T: U \rightarrow V$ be a linear map. The range of T is the set

$R(T) = \{T(u) \in V / u \in U\}$.

It is denoted as kerT.

Example:- Let $T: V_3 \rightarrow V_3$ be a linear map define by $T(x_1, x_2, x_3) = (x_1, x_2, 0)$ Find N(T) & R(T) (OR) Find the range and kernel of T.

Solution:- Here ,$R(T) = \{(x_1, x_2, 0) / x_1, x_2 \in R\}$

R(T) is x_1x_2 plane.
T is not onto.
Since R(T) is a subset of co domain V_3.
T is not one-one.
Since different vectors (1,0,2) and (1,0,5) have the same image (1,0,0).
N(T) = x_3-axis.
Since any vector (0,0, x_3) on the x_3-axis will be taken onto zero to vector of V_3.

Example:- Let T: U$\rightarrow$U be an identity linear map then find N(T) & R(T).
(OR) Find the range and kernel of T.

Solution:- Here T: U$\rightarrow$U be an identity linear map.
i.e. T(u) = u for u$\in$U.
This is one – one and onto linear map.
$\therefore$ R(T) = U and N(T) = 0

Example:- Let T: U$\rightarrow$U be zero linear map then find N(T) & R(T).
(OR) Find the range and kernel of T.

Solution:- Here T: U$\rightarrow$U be zero linear map.
i.e. T(u) = 0 for u$\in$U.
This is not one – one and onto linear map.
$\therefore$ R(T) = 0 and N(T) = U

Example:-Let T: $V_3\rightarrow V_2$ be a linear map define by T (x_1, x_2, x_3) = (x_1 - x_2, x_1+x_3) then find N(T) & R(T) (OR) Find the range and kernel of T.

. **Solution**:- Here T: $V_3\rightarrow V_2$ be a linear map define by T (x_1,x_2,x_3) = (x_1 - x_2, x_1+x_3)
Let (a, b)$\in V_2$ such that T (x_1,x_2,x_3) = (a, b)
$\therefore$ (x_1 - x_2, x_1+x_3) = (a, b)
$\therefore$ x_1 - x_2 = a , x_1+x_3 = b
Solving these equation then we get
$x_2 = x_1 –$ a, x_3 = b - x_1
Hence T (x_1, x_1 – a, b - x_1) = (a, b)
$\therefore$R(T) = V_2 ($\because$ every vector (a, b)$\in V_2$ in R(T))
So this is onto map.
Now for kernel of T
T (x_1, x_2, x_3) = (0, 0)
$\therefore$ (x_1 - x_2, x_1+x_3) = (0, 0)
$\therefore$ x_1 - x_2 = 0 , x_1+x_3 = 0
Solving these equation then we get
$\therefore x_1 = x_2 = -x_3$
i.e. all vectors of the form (x_1,x_1,-x_1) will be mapped into zero.
$\therefore$ N(T) = { x_1 (1,1,-1) / x_1 any scalar} = [(1,1,-1)]

Example:- Let T: $V_2\rightarrow V_2$ be a linear map define by T (x_1, x_2) = (x_1, - x_2) then find N(T) & R(T) (OR) Find the range and kernel of T.

. **Solution**:- **Solution**:- Here T: $V_2\rightarrow V_2$ be a linear map define by T (x_1,x_2) = (x_1, - x_2)
Let (a, b)$\in V_2$ such that T (x_1,x_2) = (a, b)

$\therefore$ $(x_1, -x_2) = (a, b)$

$\therefore$ $x_1 = a$, $-x_2 = b$

Solving these equation then we get

$x_1 = a$, $x_2 = -b$

Hence T $(a, -b) = (a, b)$

$\therefore R(T) = V_2$ $(\because$ every vector $(a, b) \in V_2$ in R(T))

So this is onto map.

Now for kernel of T

T $(x_1, x_2) = (0, 0)$

$\therefore$ $(x_1, -x_2) = (0, 0)$

$\therefore$ $x_1 = 0$, $x_2 = 0$

$\therefore$ N(T) = (0,0)}

Example:- Let the map D: $\mathcal{C}^{(1)}(a,b) \rightarrow \mathcal{C}^{(1)}(a,b)$ define by D $(f) = f'$ linear map. Where is the derivative of f. then find N(T) & R(T). (OR) Find the range and kernel of T.

Solution:- Since every continuous function g on (a, b) possesses an antiderivative. hence D is an onto map.

$\therefore R(D) = \mathcal{C}^{(1)}(a,b)$.

And N(D) is the set of all constant functions in $\mathcal{C}^{(1)}(a,b)$.

Example:- Let the map D: $\mathcal{C}^{(1)}(a,b) \rightarrow R$ define by $\mathcal{J}(f) = \int_a^b f(x)dx$ linear map. Where is the derivative of f. then find N(T) & R(T). (OR) Find the range and kernel of T.

Solution:- Since every real number can be obtained as the algebraic area under some curve $y = f(x)$ from a to b.

hence D is an onto map.

$\therefore R(D) = R$.

And it is difficult to say anything about kernel i.e. N(D).

Note :- From above example we see that if T is one-one when N(T) is the zero subspace and conversely.

Theorem:- Let T: $U \rightarrow V$ be a linear map. Then

(a) R(T) is a subspace of V.

(b) N(T) is subspace of U.

(c) T is one-one iff N(T) is the zero subspace, $\{0_U\}$, of U.

(d) If $[u_1, u_2, \ldots, u_n] = U$, then $[T(u_1), T(u_2), \ldots, T(u_n)]$

(e) If U is finite- dimensional, then $\dim R(T) \leq \dim U$.

Proof:- (a) we want to prove that R(T) is a subspace of V.

For this , let $v_1, v_2 \in$ R(T) such that $T(u_1) = v_1$ and $T(u_2) = v_2$ for $u_1, u_2 \in U$.

Now $v_1 + v_2 = T(u_1) + T(u_2) = T(u_1 + u_2)$ [$\because$ T: $U \rightarrow V$ be a linear map]

Since U is a vector space .

$\therefore$ $u_1 + u_2 \in U$

And T: U $\rightarrow$ V be a linear map

$\therefore$ $T(u_1 + u_2) = v_1 + v_2 \in R(T)$

Similarly , $\alpha v_1 \in R(T)$ then $\alpha v_1 = \alpha T(u_1) = T(\alpha u_1) \in R(T)$.

Thus R(T) is a subspace of V.

(b) we want to prove that N(T) is a subspace of U.

For this , let $u_1, u_2 \in N(T)$ such that $T(u_1) = 0_v$ and $T(u_2) = 0_v$ for $u_1, u_2 \in U$.

Now $T(u_1 + u_2) = T(u_1) + T(u_2) = 0_v$ [$\because$ T: U $\rightarrow$ V be a linear map]

$\therefore$ $T(u_1 + u_2) = 0_v$

$\therefore$ $u_1 + u_2 \in N(T)$

Similarly , for any scalar α $T(\alpha u_1) = \alpha T(u_1) = \alpha 0_v = 0_v \in N(T)$.

$\therefore$ $\alpha u_1 \in N(T)$

Thus N(T) is a subspace of V.

(c) Suppose T is one-one.

We want to prove that N(T) is the zero subspace, $\{0_U\}$, of U.

Since T is one-one then $T(u) = T(v) \Rightarrow u = v$

If $u \in N(T)$ then $T(u) = 0_v = T(0_U)$.

$\therefore u = 0_U$

i.e. no nonzero vector u of U can belong to N(T).

Since 0_U in any case belongs to N(T).

i.e. N(T) contains only 0_U and nothing else.

Hence, N(T) is the zero subspace, $\{0_U\}$, of U.

Conversely,

Suppose $N(T) = \{0_U\}$

We want to prove that T is one-one.

i.e. We want to prove that $T(u) = T(v) \Rightarrow u = v$

suppose $T(u) = T(v)$

then $T(u-v) = T(u) - T(v) = 0_V$.

$\therefore$ $u - v \in N(T) = \{0_U\}$

$\therefore$ $u - v = 0_U$.

i.e. u = v

i.e. T is one-one.

(d) let $[u_1, u_2, \ldots, u_n] = U$

$u \in U$

then u can be expressed as a linear combination of vectors $u_1, u_2, \ldots, u_n$.

The $T(u_1), T(u_2), \ldots, T(u_n)$ are in R(T).

So $[T(u_1), T(u_2), \ldots, T(u_n)] \subset R(T)$.

Let $v \in R(T)$.

Then there exists a vector $u \in U$ such that $T(u) = v$.

Since $u \in U = [u_1, u_2, \ldots, u_n]$, we have

$u = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n$.

$v = T(u) = T(\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n)$

$= \alpha_1 T(u_1) + \alpha_2 T(u_2) + \alpha_3 T(u_3) + \ldots + \alpha_n T(u_n)$.

So $v \in [T(u_1), T(u_2), \ldots, T(u_n)]$.

This proves that $R(T) = [T(u_1), T(u_2), \ldots, T(u_n)]$.

(e) Let U be finite dimensional and dim U = n.
So there can be at most n LI vectors in U.
Let $\{u_1,u_2,\ldots,u_n\}$ be the basis of U.
Then $R(T) = [T(u_1),T(u_2),\ldots,T(u_n)]$
So that there can't be more than n LI vectors inR(T).
So $\dim R(T)= n \leq \dim U$.

Definition:- Rank of T:-
Let $T: U \to V$ be a linear map. Then If R(T) is finite- dimensional, the dimension of R(T) (i.e.dimR(T)) is called the rank of T and is denoted by r(T).

Definition:- nullity of T:-
Let $T: U \to V$ be a linear map. Then If N(T) is finite- dimensional, the dimension of N(T) (i.e.dimN(T)) is called the nullity of T and is denoted by n(T).

Rank and Nullity

Theorem:- Let $T: U \to V$ be a linear map. Then
(a) If T is one-one and $u_1,u_2,\ldots,u_n$. are linearly independent vectors of U, then $T(u_1),T(u_2),\ldots,T(u_n)$ are LI.
(b)If $v_1,v_2,\ldots,v_n$ are linearly independent vectors of R(T) and $u_1,u_2,\ldots,u_n$ are vectors of U such that $T(u_1) = v_1, T(u_2) = v_2, \ldots,T(u_n) = v_n$ then $u_1,u_2,\ldots,u_n$. are linearly independent.

Proof:- (a) Let T is one-one and $\{u_1,u_2,\ldots,u_n\}$are linearly independent vectors in U.
We want to prove that $T(u_1),T(u_2),\ldots,T(u_n)$ are LI.
Consider $\alpha_1 T(u_1)+\alpha_2 T(u_2)+\alpha_3 T(u_3)+ \ldots+\alpha_n T(u_n) = 0_v$
$\therefore T(\alpha_1 u_1+\alpha_2 u_2+\ldots+\alpha_n u_n) = 0_v$ [$\because$ T is linear map]
Also $T(0_u) = 0_v$
Since T is one-one is given.
$\therefore \alpha_1 u_1+\alpha_2 u_2+\ldots+\alpha_n u_n= 0_u$
Since $u_1,u_2,\ldots,u_n$. are linearly independent vectors of U is given.
$\therefore \alpha_1=\alpha_2=\ldots=\alpha_n= 0$
Thus $T(u_1),T(u_2),\ldots,T(u_n)$ are LI.

(b) Since $v_1,v_2,\ldots,v_n$ are linearly independent vectors in V
and $T(u_1) = v_1, T(u_2) = v_2, \ldots,T(u_n) = v_n$ where $u_1,u_2,\ldots,u_n \in U$ is given.
We want to prove that $u_1,u_2,\ldots,u_n$. are linearly independent.
Consider $\alpha_1 u_1+\alpha_2 u_2+\ldots+\alpha_n u_n = 0$
$\therefore T(\alpha_1 u_1+\alpha_2 u_2+\ldots+\alpha_n u_n) = 0_v$ [$\because$ T is linear map]
$\therefore \alpha_1 T(u_1)+\alpha_2 T(u_2)+\alpha_3 T(u_3)+ \ldots+\alpha_n T(u_n) = 0_v$
Since $T(u_1),T(u_2),\ldots,T(u_n)$ are LI
$\therefore \alpha_1=\alpha_2=\ldots=\alpha_n= 0$
Hence $u_1,u_2,\ldots,u_n$. are linearly independent vectors in U.

Theorem:- (Rank and Nullity Theorem):
Let $T: U \to V$ be a linear map and U a finite dimensional vector space. Then prove that

dim R(T) + dim N(T) = dim U.

i.e r(T) + n(T) = dim U

(or) rank + nullity = dimension of the domain space.

Proof:- N(T) is a subspace of a finite dimensional vector space U. Then N(T) must be finite dimensional.

Let dim N(T) = n(T) = n and dim U = p .

So, $n \le p$

Let the basis for N(T) be { $u_1, u_2, \ldots, u_n$}.

$\therefore$ { $u_1, u_2, \ldots, u_n$}are linearly independent vectors in N(T)

$\therefore$ {$u_1, u_2, \ldots, u_n$}are linearly independent vectors in U.

Now extend this set of n linearly independent vectors of U to the basis for u.

So we find the vectors $u_{n+1}, u_{n+2}, \ldots, u_p$

So that the enlarged set { $u_1, u_2, \ldots, u_n$, $u_{n+1}, u_{n+2}, \ldots, u_p$ } is a basis for U.

Since this set of p vectors generate vector space U.

R(T) =[$T(u_1), T(u_2), \ldots, T(u_p)$]

But $u_i \in N(T)$, $i = 1, 2, 3, \ldots, n$.

Hence T(u) = 0_v, $i = 1, 2, 3, \ldots, n$.

$\therefore$ R(T) = [$T(u_{n+1}), T(u_{n+2}), \ldots, T(u_p)$]

Now we shall prove that A = { $T(u_{n+1}), T(u_{n+2}), \ldots, T(u_p)$} is basis for R(T).

Since we already proved that R(T) = [$T(u_{n+1}), T(u_{n+2}), \ldots, T(u_p)$]

So we have only prove that A is LI set.

Let us consider $\alpha_{n+1} T(u_{n+1}) + \alpha_{n+2} T(u_{n+2}) + \ldots + \alpha_p T(u_p) = 0$

$\therefore T[\alpha_{n+1} u_{n+1} + \alpha_{n+2} u_{n+2} + \ldots + \alpha_p u_p] = 0$

Since T is linear

$\alpha_{n+1} u_{n+1} + \alpha_{n+2} u_{n+2} + \ldots + \alpha_p u_p \in N(T)$.

But N(T) has a basis {$u_1, u_2, \ldots, u_n$}

so $\alpha_{n+1} u_{n+1} + \alpha_{n+2} u_{n+2} + \ldots + \alpha_p u_p$ which is the elements of N(T) can be expressed as a linear combination of basis {$u_1, u_2, \ldots, u_n$} of N(T).

$\therefore$ there exists scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$\alpha_{n+1} u_{n+1} + \alpha_{n+2} u_{n+2} + \ldots + \alpha_p u_p = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n$

$\therefore \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n + (-1)\alpha_{n+1} u_{n+1} + (-1)\alpha_{n+2} u_{n+2} + \ldots + (-1)\alpha_p u_p = 0$

Since {$u_1, u_2, \ldots, u_p$} is a basis for t vector space U this set is LI.

So $\alpha_{n+1} = \alpha_{n+2} = \ldots = \alpha_p = 0$

This prove that set A is LI.

$\therefore$ A = { $T(u_{n+1}), T(u_{n+2}), \ldots, T(u_p)$} is basis for R(T).

Dim R(T) = number of elements in basis A

= p – n

= dim U - dim N(T)

Hence rank + nullity = dimension of the domain space.

Example:- Prove that the linear map T : $V_3 \to V_3$ define by T (e_1) = $e_1 - e_2$, T (e_2) = $2e_2 + e_3$ and T(e_3) = $e_1 + e_2 + e_3$ is neither one-one nor onto.

Solution:- Here R(T) = [T (e_1), T (e_2) , T(e_3)]

= [$e_1 - e_2$, $2e_2 + e_3$, $e_1 + e_2 + e_3$]

$=[\, e_1 - e_2 \,,\, 2e_2 + e_3]$

Since $e_1 + e_2 + e_3$ is linear combination of $e_1 - e_2$, $2e_2 + e_3$

R(T) has dimension 2.

$R(T) \neq V_3$

$\therefore$ T is not onto.

Since N(T) consists those vectors $(x_1 , x_2 , x_3) \in V_3$ such that T $(x_1 , x_2 , x_3) = 0$.

i.e $T(x_1e_1 + x_2\, e_2 + x_3\, e_3) = 0$

$\therefore x_1\, T(e_1) + x_2\, T(e_2) + x_3\, T(e_3) = 0$

$\therefore x_1\, (e_1 - e_2) + x_2\, (2e_2 + e_3\,) + x_3\, (e_1 + e_2 + e_3) = 0$

$\therefore x_1 + x_3 = 0$, $-x_1 + 2\, x_2 + x_3 = 0$ and $x_2 + x_3 = 0$

Solving these equation then we get $x_1 = x_2 = -x_3$

$\therefore$ N(T) = { $(x_1 , x_1 , -x_1)$/ x_1 an arbitrary scalar}= [(1, 1,-1)].

$\therefore$ N(T) is not the zero subspace of V_3.

Hence T is not one-one.

Example:- Let linear map T : $V_4 \rightarrow V_3$ define by T (e_1) =(1, 1, 1), T (e_2) = (1, -1, 1) , T(e_3) = (1, 0, 0) and T(e_4) = (1, 0, 1) then verify that r(T) + n(T) = dim U(=V_4) = 4.

Solution:- Here R(T) = [T (e_1), T (e_2) , T(e_3), T(e_4)]

$\therefore$ R(T) = [(1, 1, 1), (1, -1, 1) , (1, 0, 0) , (1, 0, 1)]

(1, 1, 1), (1, -1, 1) , (1, 0, 0) (1, 0, 1) is LD, because a set of four vectors of V_3 (dim V_3 = 3)is always LD.

Now find that (1, 0, 1) = $\frac{1}{2}$ (1, 1, 1) + $\frac{1}{2}$ (1, -1, 1) +0 (1, 0, 0)

Hence we discard the vector (1, 0, 1) so that

R(T) = [(1, 1, 1), (1, -1, 1) , (1, 0, 0)]

Now check R(T) is LI.

Let $\alpha_1 , \alpha_2 , \alpha_3 \in$ R such that α_1 (1, 1, 1)+ α_2 (1, -1, 1)+ α_3 (1, 0, 0) = 0

$\therefore (\alpha_1 + \alpha_2 + \alpha_3 , \alpha_1 - \alpha_2 , \alpha_1 + \alpha_2) = 0$

$\therefore \alpha_1 + \alpha_2 + \alpha_3 = 0, \alpha_1 - \alpha_2 = 0, \alpha_1 + \alpha_2 = 0$

From above equations we get

$\alpha_1 = \alpha_2 = \alpha_3 = 0$

Hence {(1, 1, 1), (1, -1, 1) , (1, 0, 0)} is LI.

$\therefore$ dim R(T) = r(T) = 3

Now we find N(T).

Let us suppose that T(u) = 0 for $u \in V_4$ where u = (x_1 , x_2 , x_3, x_4),

Now u $= x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4$

$\therefore$ T(u) = $T(x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4) = 0$

$\therefore x_1\, T(e_1) + x_2\, T(e_2) + x_3\, T(e_3) = 0$

$\therefore x_1$ [(1, 1, 1)+ x_2 (1, -1, 1) + x_3 (1, 0, 0) ++ x_4 (1, 0, 1) = 0

$\therefore x_1 + x_2 + x_3 + x_4 = 0$, $x_1 - x_2 = 0$ and $+ x_1 + x_2 + x_4 = 0$

Solving these equation then we get $x_1 = x_2 = -x_4/2$, $x_3 = 0$

$\therefore$ N(T) = { $(x_1 , x_1 , 0, -2x_1)$/ x_1 an arbitrary scalar}= [(1, 1,0,-2)].

$\therefore$ dim N(T) = n(T) =1

Hence r(T) + n(T) = 3 +1 = 4 = dim U(=V_4)

Inverse of a linear transformation

Definition:- Nonsingular or Isomorphism :

A linear map $T : U \to V$ is said to be nonsingular if it is one – one and onto. Such a map is also called an isomorphism.

Definition:- Inverse function:

Any function is called inverse function iff it is one – one and onto.

Note: a linear transformation is Nonsingular iff it has an inverse.

Example:- Define a linear map $T : V_2 \to V_2$ by $T(x_1 , x_2) = (x_1 , -x_2)$. Prove that this map is Nonsingular.

Solution:- Here the linear map $T : V_2 \to V_2$ define by $T(x_1 , x_2) = (x_1 , -x_2)$

Now for N(T)

$T(x_1 , x_2) = 0$

$\therefore (x_1 , -x_2) = 0$

$\therefore x_1 = 0$ and $x_2 = 0$

$N(T) = 0$

$\therefore$ This linear map T is one – one.

Now for R(T)

For every $(y_1 , y_2) \in V_2$ then there exists $(x_1 , x_2) \in V_2$ such that

$T(x_1 , x_2) = (y_1 , y_2)$

$\therefore (x_1 , -x_2) = (y_1 , y_2)$

$\therefore x_1 = y_1$ and $x_2 = y_2$

$\therefore R(T) = V_2$

This linear map T is onto

$\therefore T^{-1}(y_1 , y_2) = (y_1 , -y_2)$.

$\therefore$ it has inverse

$\therefore$ This map is Nonsingular.

Example:- Prove that a linear map $T: U \to U$ define by $T(u) = u$ is Nonsingular.

(OR)Prove that an identity linear map is one-one and onto.

(OR) Find the inverse of a linear map $T: U \to U$ define by $T(u) = u$.

Solution:- Here the linear map $T : U \to U$ define by $T(u) = (u)$

Now for N(T)

$T(u) = 0$

$\therefore u = 0$

$\therefore N(T) = 0$

$\therefore$ This linear map T is one – one.

Now for R(T)

For every $(y) \in U$ then there exists $(u) \in U$ such that

$T(u) = (y)$

$\therefore (u) = (y)$

$\therefore R(T) = U$

This linear map T is onto

$\therefore T^{-1}(y) = (y)$.

$\therefore$ it has inverse

$\therefore$ This map is Nonsingular.

Example:- Prove that a linear map T : $\mathscr{P}_2 \to V_3$ define by T $(\alpha_0 + \alpha_1 x + \alpha_2 x^2) = (\alpha_0, \alpha_1, \alpha_2)$ is Nonsingular. OR Prove that a linear map T : $\mathscr{P}_2 \to V_3$ define by T $(\alpha_0 + \alpha_1 x + \alpha_2 x^2) = (\alpha_0, \alpha_1, \alpha_2)$ is isomorphism.

Solution:- Here the linear map T : $\mathscr{P}_2 \to V_3$ define by T $(\alpha_0 + \alpha_1 x + \alpha_2 x^2) = (\alpha_0, \alpha_1, \alpha_2)$

Now for N(T)

$T(\alpha_0 + \alpha_1 x + \alpha_2 x^2) = 0$

$\therefore (\alpha_0, \alpha_1, \alpha_2) = 0$

$\therefore \alpha_0 = \alpha_1 = \alpha_2 = 0$

$\therefore N(T) = 0$

$\therefore$ This linear map T is one – one.

Now for R(T)

For every $(\beta_1, \beta_2, \beta_3) \in V_3$ then there exists $(\beta_1 + \beta_2 x + \beta_3 x^2) \in \mathscr{P}_2$ such that

$T(\beta_1 + \beta_2 x + \beta_3 x^2) = (\beta_1, \beta_2, \beta_3)$

$\therefore (\beta_1, \beta_2, \beta_3)$ is image of T.

$\therefore R(T) = V_3$

This linear map T is onto

$\therefore T^{-1}(\beta_1, \beta_2, \beta_3) = (\beta_1 + \beta_2 x + \beta_3 x^2)$.

$\therefore$ it has inverse

$\therefore$ This map is Nonsingular.

Theorem:- Let T : $U \to V$ be nonsingular linear map. Then $T^{-1} : V \to U$ is a linear, one – one and onto map.

Proof:- First to prove that T^{-1} is linear.

Let $v_1, v_2 \in V$.

Let $T^{-1}(v_1) = u_1$ and $T^{-1}(v_2) = u_2$ for $u_1, u_2 \in U$.

Since T is nonsingular linear map

$\therefore$ T is one – one and onto map.

$\therefore$ u_1 and u_2 exists uniquely.

$\therefore v_1 = T(u_1)$ and $v_2 = T(u_2)$

$\therefore v_1 + v_2 = T(u_1) + T(u_2) = T(u_1 + u_2)$ [$\because$ T is linear]

$\therefore T^{-1}(v_1 + v_2) = u_1 + u_2 = T^{-1}(v_1) + T^{-1}(v_2)$

$\therefore$ we get $T^{-1}(v_1 + v_2) = T^{-1}(v_1) + T^{-1}(v_2)$

Also $\alpha v_1 = \alpha\, T(u_1) = T(\alpha u_1)$

$\therefore T^{-1}(\alpha v_1) = \alpha u_1 = \alpha T^{-1}(v_1)$

i.e. $T^{-1}(\alpha v_1) = \alpha T^{-1}(v_1)$

$\therefore T^{-1}$ is linear.

Now we want to prove that T^{-1} is one-one.

Let $v_1, v_2 \in V$ such that $T^{-1}(v_1) = T^{-1}(v_2) = u$ say for $u \in U$.

$\Rightarrow \quad v_1 = v_2 = T(u)$

Since image of u under T is unique.

$\therefore\ v_1 = v_2$

Thus we get $T^{-1}(v_1) = T^{-1}(v_2) \Rightarrow \quad v_1 = v_2$

$\therefore\ T^{-1}$ is one-one.

Now we want to prove that T^{-1} is onto.

Given any element $u \in U$ then there exists an element $v \in V$ such that

$T(u) = v$.

$\therefore u = T^{-1}(v)$

This show that T^{-1} is onto.

$\therefore\ T^{-1} : V \to U$ is a linear, one – one and onto map.

Example:- Check that a linear map $T: U \to U$,where U is vector space, define by $T(x) = T(\{x_1, x_2, x_{3,\dots,} x_n, \dots\}) = \{x_2, x_{3,\dots,} x_n, \dots\}$ is Nonsingular or not. Also check the inverse of this linear map is exist or not.

Solution:- Here the linear map $T : U \to U$ define by

$T(\{x_1, x_2, x_{3,\dots,} x_n, \dots\}) = \{x_2, x_{3,\dots,} x_n, \dots\}$

Now for N(T)

$T(x) = 0$

$\therefore\ T(\{x_1, x_2, x_{3,\dots,} x_n, \dots\}) = 0$

$\therefore\ \{x_2, x_{3,\dots,} x_n, \dots\} = 0$

Here let $x_1 = z$ which may not be zero.

$\therefore N(T) = \{z, 0, 0, 0, \dots\dots\}$

$\therefore N(T) \neq 0$

$\therefore$ This linear map T is not one – one.

Now for R(T)

For every $(y_1, y_2, y_{3,\dots,} y_n, \dots) \in U$ then there exists $(z, y_1, y_2, y_{3,\dots,} y_n, \dots) \in U$ such that $T(\{z, y_1, y_2, y_{3,\dots,} y_n, \dots\}) = (y_1, y_2, y_{3,\dots,} y_n, \dots)$

$\therefore \{z, y_1, y_2, y_{3,\dots,} y_n, \dots\}$ is pre image of $\{y_1, y_2, y_{3,\dots,} y_n, \dots\}$

$\therefore R(T) = U$

This linear map T is onto

Thus we get T is onto but not one-one.

$\therefore$ This map is not nonsingular.

$\therefore$ Also T has not inverse.

Example:- Check that a linear map $T: U \to U$,where U is vector space, define by $T(x) = T(\{x_1, x_2, x_{3,\dots,} x_n, \dots\}) = \{0, x_1, x_2, x_{3,\dots,} x_n, \dots\}$ is Nonsingular or not. Also check the inverse of this linear map is exists or not.

Solution:- Here the linear map $T : U \to U$ define by

$T(\{x_1, x_2, x_{3,\dots,} x_n, \dots\}) = \{0, x_1, x_2, x_{3,\dots,} x_n, \dots\}$

Now for N(T)

$T(x) = 0$

$\therefore\ T(\{x_1, x_2, x_{3,\dots,} x_n, \dots\}) = 0$

$\therefore\ \{0, x_1, x_2, x_{3,\dots,} x_n, \dots\} = 0$

$\therefore N(T) = \{0, 0, 0, 0, \dots\dots\}$

$\therefore$ N(T) = 0

$\therefore$ This linear map T is one – one.

Now for R(T)

let $(1, 1, 1_{,\dots,} 1, \dots) \in U$ has no pre image in U

$\therefore$ R(T) $\neq$ U

This linear map T is not onto

Thus we get T is not onto but one-one.

$\therefore$ This map is not nonsingular.

$\therefore$ Also T has not inverse.

Theorem:- If U and V are finite dimensional vector spaces of the same dimension, then a linear map T: U$\to$ V is one-one iff it is onto.

Proof:- T is one –one $\Leftrightarrow$ N(T) = $\{0_v\}$

$\Leftrightarrow$ n(T) = 0

$\Leftrightarrow$ r(T) = dim U = dim V

[$\because$ By Rank and Nullity Theorem i.e r(T) + n(T) = dim U]

$\Leftrightarrow$ T is onto.

Example:- Show that the linear map T : $V_3 \to V_3$ defined by

$T(x_1, x_2, x_3) = (x_1 + x_2 + x_3, x_2 + x_3, x_3)$ is nonsingular and find its inverse.

Solution:- Here the linear map T : $V_3 \to V_3$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2 + x_3, x_2 + x_3, x_3)$

Now for N(T)

$T(x_1, x_2, x_3) = 0$

$\therefore (x_1 + x_2 + x_3, x_2 + x_3, x_3) = 0$

$x_1 + x_2 + x_3 = 0, x_2 + x_3 = 0, x_3 = 0$

$\therefore x_1 = 0, x_2 = 0$ and $x_3 = 0$

N(T) = 0

$\therefore$ This linear map T is one – one.

Since the dimension of domain space and dimension of co domain space are equal.

$\therefore$ T is onto.

Thus we get T is one-one and onto.

Hence T is nonsingular and the inverse of T exists. i.e.T^{-1} exists.

Now we derive the formula for T^{-1}.

Let $T^{-1}(y_1, y_2, y_3) = (x_1, x_2, x_3)$.___________(1)

$\therefore (y_1, y_2, y_3) = T(x_1, x_2, x_3)$

$\therefore (y_1, y_2, y_3) = (x_1 + x_2 + x_3, x_2 + x_3, x_3)$

$\therefore y_1 = x_1 + x_2 + x_3, y_2 = x_2 + x_3, y_3 = x_3$

Solving these equation then we get

$x_1 = y_1 - y_2$, $x_2 = y_2 - y_3$ and $x_3 = y_3$

put the values of x_1, x_2, x_3 in equation (1) then we get

$T^{-1}(y_1, y_2, y_3) = (y_1 - y_2, y_2 - y_3, y_3)$

Example:- Show that the linear map T : $V_3 \to V_3$ defined by $T(e_1) = e_1 + e_2$, $T(e_2) = e_2 + e_3$ and $T(e_3) = e_1 + e_2 + e_3$ is nonsingular and find its inverse.

Solution:- First we find the value of T.

Let $(x_1, x_2, x_3) \in V_3$ such

$T(x_1, x_2, x_3) = T(x_1e_1 + x_2 e_2 + x_3 e_3)$
$= x_1 T(e_1) + x_2 T(e_2) + x_3 T(e_3)$
$= x_1 (e_1 + e_2) + x_2 (e_2 + e_3) + x_3(e_1 + e_2 + e_3)$
$= x_1 (1,1,0) + x_2 (0,1,1) + x_3(1,1,1)$
[$\because$ Where $e_1= (1,0,0)$, $e_2 = (0,1,1)$ and $e_3= (0,1,1)$]
$= (x_1 + x_3, x_1+ x_2 +x_3, x_2+ x_3)$

i.e. $T(x_1, x_2, x_3) = (x_1 + x_3, x_1+ x_2 +x_3, x_2+ x_3)$

Now for N(T)

$T(x_1, x_2, x_3) = 0$
$(x_1 + x_3, x_1+ x_2 +x_3, x_2+ x_3)= 0$
$x_1 + x_3 = 0, x_1 + x_2+ x_3 = 0, x_2 +x_3 = 0,$
$\therefore x_1 = 0, x_2 = 0$ and $x_3 = 0$
$N(T) = 0$
$\therefore$ This linear map T is one – one.

Since the dimension of domain space and dimension of co domain space are equal.
$\therefore$ T is onto.
Thus we get T is one-one and onto.
Hence T is nonsingular and the inverse of T exists. i.e.T^{-1} exists.
Now we derive the formula for T^{-1}.
Let $T^{-1}(y_1, y_2, y_3) = (x_1, x_2, x_3)$.___________(1)
$\therefore (y_1, y_2, y_3) = T(x_1, x_2, x_3)$
$\therefore (y_1, y_2, y_3) = (x_1 + x_3, x_1+ x_2 +x_3, x_2+ x_3)$
$\therefore y_1 = x_1 + x_3, y_2 = x_1+ x_2 +x_3, y_3= x_2+ x_3$
Solving these equation then we get
$x_1 = y_2 - y_3$, $x_2 = y_2 - y_1$ and $x_3 = y_1 - y_2 + y_3$
put the values of x_1, x_2, x_3 in equation (1) then we get
$T^{-1}(y_1, y_2, y_3) = (y_2 - y_3, y_2 - y_1, y_1 - y_2 + y_3)$

Consequences of rank nullity theorem

Definition:-**Isomorphic**:-

Two vector spaces U and V are said to be isomorphic if there exists an isomorphism from U to V.
If U and V are isomorphic then we write $U \approx V$.

Theorem:- Every real (complex) vector space of dimension p is isomorphic to $V_p(V_p^c)$

Proof:- Let U be a real vector space of dimension p.
Let $B = \{u_1, u_2, u_3, \ldots, u_p\}$ an ordered basis for U.
Let $u \in U$
$\therefore u = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_p u_p$
i.e u is linear combination of set B.
$\therefore$ the coordinate vector of U relative to B is $\{\alpha_1, \alpha_2, \ldots, \alpha_p\}$
Now define a mapping $T : U \to V_p$ by $T(u) = (\alpha_1, \alpha_2, \ldots, \alpha_p)$
We want to prove that T is linear map.
Let $u, v \in U$ where $u = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_p u_p$ and $v = \beta_1 u_1 + \beta_2 u_2 + \ldots + \beta_p u_p$

$\therefore$ $u + v = (\alpha_1 + \beta_1)u_1 + (\alpha_2 + \beta_2)u_2 + \ldots + (\alpha_p + \beta_p)u_p$

Now $T(u + v) = T((\alpha_1 + \beta_1)u_1 + (\alpha_2 + \beta_2)u_2 + \ldots + (\alpha_p + \beta_p)u_p)$

$= ((\alpha_1 + \beta_1), (\alpha_2 + \beta_2), \ldots, (\alpha_p + \beta_p))$ ________(1) [$\because$ by def^n of T]

And $T(u) + T(v) = (\alpha_1, \alpha_2, \ldots, \alpha_p) + (\beta_1, \beta_2, \ldots, \beta_p)$

$= ((\alpha_1 + \beta_1), (\alpha_2 + \beta_2), \ldots, (\alpha_p + \beta_p))$ ________(2)

From (1) and (2)

$T(u + v) = T(u) + T(v)$ ___________(3)

Similarly if α is any scalar and $u \in U$ then it can be proved that

$T(\alpha u) = \alpha T(u)$ _____________(4)

From (3) and (4)

Hence T is linear map.

Now we want to prove that T is one –one.

Let $u \in N(T)$

$\therefore$ $T(u) = 0_{vp}$

$\therefore$ $(\alpha_1, \alpha_2, \ldots, \alpha_p) = 0$

$\therefore$ $\alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_p = 0$

$\therefore u = 0_u$

N(T) is zero subspace of U.

$\therefore$ T is one –one.

Also T is onto.

Hence T is an isomorphism from U to V_p.

$\therefore$ U and V_p are isomorphic i.e $U \approx V_p$.