SEM-V
MAT 301: Linear Algebra- II (Theory)
Unit-1

Definition:- Composition of linear maps:-

Let $T : U \to V$ and $S : V \to W$ be two linear maps. Then the composition $SoT : U \to W$ is defined by $SoT(u) = S(T(u))$ for all $u \in U$. Here SoT is called the composition of S and T.

Example:- Show that SoT is a linear map.

Solution:- Let $u_1, u_2 \in U$ and α is any scalar. Then

(i) $SoT(u_1 + u_2) = S(T(u_1 + u_2))$ [$\because$ by definition of composition]
$= S(T(u_1) + T(u_2))$ [$\because$ T is linear map]
$= S(T(u_1)) + S(T(u_2))$ [$\because$ S is linear map]
$= SoT(u_1) + SoT(u_2)$ [$\because$ by definition of composition]

(ii) $SoT(\alpha u_1) = S(T(\alpha u_1))$ [$\because$ by definition of composition]
$= S(\alpha T(u_1))$ [$\because$ T is linear map]
$= \alpha S(T(u_1))$ [$\because$ S is linear map]
$= \alpha SoT(u_1)$ [$\because$ by definition of composition]

From (i) & (ii) SoT is a linear map
i.e. The composition of two linear map is again a linear map.

Example:- Let a linear map $T : V_3 \to V_4$ be defined by
$T(e_1) = (1, 1, 0, 0)$, $T(e_2) = (1, -1, 1, 0)$ and $T(e_3) = (0, -1, 1, 1)$, where $\{e_1, e_2, e_3\}$ is the standard basis for V_3, and let a linear map $S : V_4 \to V_2$ be defined by
$S(f_1) = (1,0)$, $S(f_2) = (1, 1)$, $S(f_3) = (1, -1)$ and $S(f_4) = (0, 1)$, where $\{f_1, f_2, f_3, f_4\}$ is the standard basis for V_4 . Then find $SoT: V_3 \to V_2$.

Solution:- Since

$SoT(e_1) = S(T(e_1)) = S(1, 1, 0, 0) = S(f_1 + f_2) = S(f_1) + S(f_2) = (1,0) + (1, 1) = (2,1)$

Now

$SoT(e_2) = S(T(e_2)) = S(1, -1, 1, 0) = S(f_1 - f_2 + f_3) = S(f_1) - S(f_2) + S(f_3) =$
$(1,0) - (1, 1) + (1, -1) = (1,-2)$

Now

$SoT(e_3) = S(T(e_3)) = S(0, -1, 1, 1) = S(-f_2 + f_3 + f_4) = -S(f_2) + S(f_3) + S(f_4) =$
$-(1, 1) + (1, -1) + (0, 1) = (0,-1)$

Note:-

- We can write ST for SoT and call it the product of S and T rather than the composition of S and T.
- If ST defined then TS need not be defined. Even if both are defined, they need not be equal. Thus the commutative law of the product is not in general satisfied. The other laws of multiplication are easily seen to hold.

Theorem:- LetT_1 ,T_2 be linear maps from U to V. Let S_1, S_2 be linear maps from V to W. P be linear maps from W to Z, where U, V, W and Z are vector spaces over the same field of scalars. Then prove that

(a) $S_1(T_1 + T_2) = S_1T_1 + S_1T_2$.

(b) $(S_1 + S_2) T_1 = S_1T_1 + S_2T_1$.

(c) $P(S_1T_1) = P(S_1)T_1$.

(d) $(\alpha S_1)T_1 = \alpha (S_1T_1) = S_1(\alpha T_1)$, where α is a scalar.

(e) $I_VT_1 = T_1$ and $T_1 I_U = T_1$

Proof :(a)

Since T_1 and T_2 be linear maps from U to V .

i.e. $T_1 : U\to V$,$T_2 : U\to V$

$\therefore$ $T_1 + T_2$ be linear maps from U to V.

i.e. $T_1 + T_2: U\to V$ be linear maps.

Also S_1 be linear maps from V to W.

i.e. $S_1 : V\to W$ be linear maps.

$\therefore$ $S_1 (T_1 + T_2)$ be linear maps from U to W.

i.e. $S_1 (T_1 + T_2): U\to W$ be linear maps and $S_1T_1 + S_1T_2$ is also defined.

$\therefore$ $S_1 (T_1 + T_2)$ and $S_1T_1 + S_1T_2$ have the same domain U.

Let $u\in U$ then

$S_1[(T_1 + T_2)](u) = S_1[(T_1 + T_2) (u)]$ [$\because$ by definition of product]

$= S_1[T_1(u) + T_2 (u)]$ [$\because$ by sum of linear map]

$= S_1(T_1(u)) + S_1 (T_2 (u))$ [$\because S_1$ is linear map]

$= (S_1T_1)(u) + (S_1T_2)(u)$ [$\because$by definition of commutative]

$=(S_1T_1 + S_1T_2)(u)$ [$\because$ by definition of sum of linear map]

This proved that $S_1(T_1 + T_2) = S_1T_1 + S_1T_2$.

Proof of (b) is similar to (a).

(c) Since $P :W \to Z$,$S_1 : V\to W$ be linear maps and $T_1 : U\to V$ be linear maps.

$\therefore$ $S_1T_1 : U\to W$ be linear maps.

$\therefore P(S_1T_1): U\to Z$ be linear maps.

$\therefore$ the domain of $P(S_1T_1)$ and $P(S_1)T_1$ is common.

Let $u\in U$ then

$[P(S_1T_1)](u) =P[(S_1T_1) (u)] = P[(S_1\{T_1 (u)\}] = \{(PS_1)T_1 (u)\} = P\{S_1)T_1 (u)\}$

Hence, images of u under the two functions are same.

$\therefore$ we get

$P(S_1T_1) = P(S_1)T_1$

(d) Proof (d) is simple.

(e) Domain of I_VT_1 = domain of T_1=U. So the functions

I_VT_1 and T are same

$(I_VT_1)(u)= I_V(T_1(u))$

Similarly $T_1I_u = T_1$.

$= T_1(u)$

$\therefore$ $I_VT_1=T_1$

Note :- We know that $T : U \rightarrow V$ be a nonsingular linear map, i.e. T is one-one and onto. Then $T^{-1} : V \rightarrow U$ exists and is linear. Further $TT^{-1} = I_V$ and $T^{-1}T = I_U$.

Theorem:- $T : U \rightarrow V$ and $S: V \rightarrow W$ be a linear maps. Then

(a) If S and T are nonsingular, then ST is nonsingular and $(ST)^{-1} = T^{-1}S^{-1}$.
(b) If ST is one-one, then T is one-one..
(c) If ST is onto, then S is onto.
(d) If ST is nonsingular, then T is one-one and S is onto.
(e) If U, V, W are of the same finite dimension and ST is nonsingular, then both S and T are nonsingular.

Proof: Since S is nonsingular. S^{-1} exists and $SS^{-1} = I_W$ and $S^{-1}S = I_v$.
Since T is nonsingular. T^{-1} exists and $TT^{-1} = I_V$ and $T^{-1}T = I_U$.
Then we have $(ST)(T^{-1}S^{-1}) = (S(T(T^{-1}S^{-1}))) = (S(TT^{-1})S^{-1}) = S(I_V S^{-1}) = SS^{-1} = I_W$.
Similarly,
$(T^{-1}S^{-1})(ST) = (T^{-1}(S^{-1}(ST)) = (T^{-1}((S^{-1}S)T)) =$ T-1()IVT) $= T^{-1}T = I_U$
Hence ST is nonsingular and $(ST)^{-1} = T^{-1}S^{-1}$.

The Space L(U, V)

Definition:- Sum of two linear maps:

Let $T: U \rightarrow V$ and $S : U \rightarrow V$ be two linear transformations. The linear map $M: U \rightarrow V$ defined by $M(u) = S(u) + T(u)$ for all $u \in U$ is called the sum of two linear map S and T.

Example:- Let $T: U \rightarrow V$ and $S : U \rightarrow V$ be two linear transformations. Then prove that $M: U \rightarrow V$ defined by $M(u) = S(u) + T(u)$ for all $u \in U$ linear map.

Solution:- Let $u_1, u_2 \in U$ then

$M(u_1 + u_2) = S(u_1 + u_2) + T(u_1 + u_2)$ [$\because$ by definition of M]
$= (S(u_1) + S(u_2)) + (T(u_1) + S(u_2))$ [$\because$ S and T linear map]
$\therefore M(u_1 + u_2) = (S(u_1) + S(u_2)) + (T(u_1) + S(u_2))$ ________(i)
And $M(u_1) + M(u_2) = (S(u_1) + S(u_1)) + (T(u_2) + S(u_2))$ [$\because$ by definition of M]
$= (S(u_1) + S(u_2)) + (T(u_1) + S(u_2))$
$\therefore M(u_1) + M(u_2) = (S(u_1) + S(u_2)) + (T(u_1) + S(u_2))$ ________(ii)
From (i) & (ii)
$M(u_1 + u_2) = M(u_1) + M(u_2)$ ________________(a)

Again let $\alpha \in R$ and $u_1 \in U$ then

$M(\alpha u_1) = S(\alpha u_1) + T(\alpha u_1)$ [$\because$ by definition of M]
$= \alpha S(u_1) + \alpha T(u_1)$ [$\because$ S and T linear map]
$= \alpha (S(u_1) + T(u_1))$
$= \alpha M(u_1)$

$\therefore M(u_1) = \alpha M(u_1)$ _________________________(b)

From (a) & (b)
M: $U \to V$ be a linear map.

Definition:- Scalar multiple of a linear map:
Let $S : U \to V$ be linear transformation and α be any scalar. Then the linear map $P: U \to V$ defined by $P(u) = \alpha(S(u))$ for all $u \in U$ is called Scalar multiple of a linear map S and α.

Example:- Let $S : U \to V$ be linear transformation and α be any scalar. Then prove that $P: U \to V$ defined by $P(u) = \alpha(S(u))$ for all $u \in U$ is linear map.

Solution:- Let $u_1, u_2 \in U$ and α be any scalar then

$P(u_1 + u_2) = \alpha(S(u_1 + u_2))$ [$\because$ by definition of P]

$= \alpha(S(u_1) + S(u_2))$ [$\because$ S is linear map]

$= \alpha(S(u_1)) + \alpha(S(u_2))$

$= P(u_1) + P(u_2)$

$\therefore P(u_1 + u_2) = P(u_1) + P(u_2)$ ________________(i)

Again λ be any scalar and $u \in U$ then

$P(\lambda u) = \alpha(S(\lambda u))$ [$\because$ by definition of P]

$= \alpha(\lambda S(u))$ [$\because$ S is linear map]

$= \lambda(\alpha(S(u)))$

$= \lambda P(u)$

$\therefore P(\lambda u) = \lambda P(u)$ __________________(ii)

From (i) & (ii)
$P: U \to V$ is linear map.

Example:- Let $T: V_3 \to V_2$ and $S : V_3 \to V_2$ be two linear transformations defined by $T(x_1, x_2, x_3) = (x_1 - x_2, x_2 + x_3)$ and $S(x_1, x_2, x_3) = (2x_1, x_2 - x_3)$ then find $(S + T)$ and $\alpha(S)$.

Solution:- Since $(S + T) : V_3 \to V_2$ is given by

$(S + T)(x_1, x_2, x_3) = S(x_1, x_2, x_3) + T(x_1, x_2, x_3)$

$= (2x_1, x_2 - x_3) + (x_1 - x_2, x_2 + x_3)$

$= (3x_1 - x_2, 2x_2)$

And $\alpha S: V_3 \to V_2$ is given by

$\alpha(S)(x_1, x_2, x_3) = \alpha(S(x_1, x_2, x_3))$

$= \alpha(2x_1, x_2 - x_3)$

$= (2\alpha x_1, \alpha(x_2 - x_3))$

Example:- Let $T: V_3 \to V_3$ and $S : V_3 \to V_3$ be two linear transformations defined by $T(e_1) = (e_1 + e_2)$, $T(e_2) = e_3$, $T(e_3) = (e_2 - e_3)$; $S(e_1) = e_3$, $S(e_2) = (2e_2 - e_3)$ and $S(e_3) = 0$ then find $(S + T)$ and $2T$

Solution:- Since $(S + T) : V_3 \to V_3$ is given by

$(S + T)(e_1) = S(e_1) + T(e_1) = e_3 + (e_1 + e_2) = e_1 + e_2 + e_3$

$(S + T)(e_2) = S(e_2) + T(e_2) = (2e_2 - e_3) + e_3 = 2e_2$

$(S + T)(e_3) = S(e_3) + T(e_3) = 0 + (e_2 - e_3) = e_2 - e_3$

And 2T: $V_3 \to V_3$ is given by

$(2T)(e_1) = 2T(e_1) = 2(e_1 + e_2)$

$(2T)(e_2) = 2T(e_2) = 2e_3$

$(2T)(e_3) = 2T(e_3) = 2(e_2 - e_3)$

Note: - The set of all linear transformations from U to V is denoted by L(U, V). Here U and V are vector spaces.

Theorem: The set L(U, V) of all linear maps from U to V together with the operations of addition and scalar multiplication as defined above is a vector space.

Proof:

We have already seen that the sum of two linear maps from U to V is again a linear map from U to V. Hence L(U, V) is closed under addition. Also a scalar multiple of a linear map is again a linear map. Hence L(U, V) is closed under the operation of scalar multiplication.

Now we define zero linear map takes any vector of U into a zero vector V.

Negative of a linear map $-T: U \to V$ is defined by $(-T)(u) = (-u)$

The following properties are the consequence of these definitions.

If S, T, R are any linear maps belonging to L(U,V) and α, β any scalars then

(i) Addition in L is commutative. i.e. S + T = T+ S

(ii)Addition in L is Associative. i.e. (S + T) + R = S+(T + R)

(iii) There exists an $0 \in L$ such that T + 0 = T. Here 0 is called identity element for addition.

(iv) For each $T \in L$ there exists $-T \in L$ such that T + (- T) = 0. Here (- T) is called Inverse element for addition

(v) $\alpha(S + T) = \alpha S + \alpha T$

(vi) $(\alpha + \beta)T = \alpha T + \beta T$

(vii) $(\alpha\beta)S = \alpha(\beta S) = \alpha\ \beta S$

(viii) 1.S = S.

Hence L(U, V) satisfied all axioms for vector space so it is vector space.

i.e. the set of all linear transformation from U to the vector space V is a vector space itself.

Operator Equations

Definition: Operator Equations:

Let $T: U \to V$ be a linear map from the vector space U to the vector space V. the equation $T(u) = v_o$,Where v_o is a fixed vector in V, is called an Operator Equation.

Note:(i) if $v_0 = 0$ i.e. $T(u) = 0_v$ then the equation is called homogenous (H) equation.

(ii) if $v_0 \neq 0_v$ i.e. $T(u) = v_o$ then the equation is called nonhomogenous (NH) equation.

(iii)The set of solutions of the equation T(u) = 0 is the kernel of T i.e. N(T).

Theorem:- Let $T: U \to V$ be a linear map. Given $v_0 \neq 0_v$ in V, the nonhomogenous (NH) equation. $T(u) = v_o$ and the associated homogenous (H) equation $T(u) = 0_v$ have the following properties:

(a) If $v_0 \notin R(T)$, then (NH) has no solution for u.

(b) If $v_0 \in R(T)$ and (H) has trivial solution, namely, $u = 0_u$, as its only solution, then, (NH) has unique solution.

(c) If $v_0 \in R(T)$ and (H) has a nontrivial solution, namely, a solution $u \neq 0_U$, then (NH) has infinite number of solutions. In this case if u_0 is a solution of (NH), then the set of all solutions of (NH) is linear variety u_0+K, where $K = N(T)$ is the set of all solutions of (H).

Proof:- (a) is obvious. Recall the definition of R (T).

(b) If $v_0 \in R(T)$, then $T(u) = v_0$ has a solution.

If $T(u) = 0_v$ has only one solution, i.e. $u = 0_U$, then $N(T) = \{0_U\}$,

i.e. T is one-one.

This means $T(u) = v_0$ cannot have more than one solution,

i.e. the solution of (NH) is unique.

(c) If $T(u) = 0_v$ has a nonzero solution , then $N(T) \neq \{0_U\}$.

Let $u_0 \in U$ be a solution of (NH) .

It exists because $v_0 \in R(T)$.

Then $T(u_0) = v_0$.

Now if $u_k \in N(T)$, then

$$\begin{aligned} T(u_0+u_k) &= T(u_0) + T(u_k) \\ &= v_0 + 0_v \\ &= v_0 \end{aligned}$$

Therefore, $u_0 + u_k$ is a solution of (NH). This is true for every $u_k \in N(T)$ and since this letter has infinite number of elements in it, (NH) also has an infinite number of solutions.

From this discussion it is obvious that $u_0 + K$, where $K = N(T)$, is contained the solution set of (NH).

Conversely, If w be any other solution of (NH) then

$T(w) = v_0 = T(v_0)$ or $T(w - u_0) = v_0$

i.e. $w - u_0 \in N(T) = K$

So w and u_0 belong to the same parallel of K, namely $u_0 + K$.

Thus, the solution set of (NH) is precisely $u_0 + K$.

Note:- $u_0 + K$ is the pre-image of v_0.

Example:- Let $D: \mathbb{C}(0, 2\pi) \to \mathbb{C}(0, 2\pi)$ be the linear differential operator .the operator equation $D(f)(x) = \sin x$.

Solution:- the associated homogeneous equation (H)is as $D(f)(x)= 0$

The solution set of this equation is the set of all constant functions.

$K = \{f / f(x) = b$ for all $x \in (0, 2\pi)$ and b a constant$\}$

One solution of $D(f)(x) = \sin x$ is the function f_0, where $f_0(x) = -\cos x$.

So the solution set is $f_0 + K$.

In other words, the set of all function g, where g(x) = -cos x + (a constant) is the solution set of D(f)(x) = sin x.

Note: To solve a nonhomogeneous operator equation (NH)
T(u) = v_0,
Where T is linear operator,
We go through three steps:
Step 1. Form the associated homogeneous equation (H)
Step 2. Get all solutions of (H). It is the kernel of T, i.e. N(T).
Step 3. Get one particular solution u_0 of (NH).
Now the complete solution of (NH) is u_0 + N(T).

Examples of solving an operator equation:

Example-1 Let T: $V_5 \rightarrow V_3$ be a linear transformation defined by $\mathrm{T}(e_1) = \frac{1}{2}f_1$, $\mathrm{T}(e_2) = \frac{1}{2}f_1$, $\mathrm{T}(e_3) = f_2$, $\mathrm{T}(e_4) = f_2$ and $\mathrm{T}(e_5) = 0$. Where $\{e_1, e_2, e_3, e_4, e_5\}$ is the standard basis for V_5 and $\{f_1, f_2, f_3\}$ is the standard basis for V_3 then solve the equation T(u) = (1, 1, 0).

Solution: Frist calculate the value of T(u) i.e. $T(x_1, x_2, x_3, x_4, x_5)$ Here u$\in V_5$.
Since T is a linear map

$$\therefore\ T(x_1, x_2, x_3, x_4, x_5) = x_1\,\mathrm{T}(e_1) + x_2\mathrm{T}(e_2) + x_3\mathrm{T}(e_3) + x_4\mathrm{T}(e_4) + x_5\mathrm{T}(e_5)$$
$$= x_1\,\frac{1}{2}f_1 + x_2\frac{1}{2}f_1 + x_3 f_2 + x_4 f_2 + x_5.0 \text{ (Put the given values)}$$
$$= \left(\frac{x_1+x_2}{2}\right)f_1 + (x_3+x_4)f_2 + 0f_3$$
$$= \left(\frac{x_1+x_2}{2}, x_3 + x_4, 0\right)(f_1, f_2, f_3)$$
$$T(x_1, x_2, x_3, x_4, x_5) = \left(\frac{x_1+x_2}{2}, x_3 + x_4, 0\right)$$

The associated homogeneous equation leads to the equations
i.e. $T(x_1, x_2, x_3, x_4, x_5) = 0$

$$\frac{x_1+x_2}{2} = 0, x_3 + x_4 = 0.$$

Solving these, we get $x_2 = -x_1$, $x_3 = -x_4$
Thus, the kernel of T is the set of all vectors of form$(x_1, -x_1, x_3, -x_3, x_5)$,
i.e. $x_1\,(1,-1,0,0,0) + x_3(0,0,1,-1,0) + x_5(0,0,0,0,1)$
$\therefore$ N(T) = $[(1,-1,0,0,0), (0,0,1,-1,0), (0,0,0,0,1)]$
Now find T(u) = (1, 1, 0).
Since T(u) = $T(x_1, x_2, x_3, x_4, x_5) = \left(\frac{x_1+x_2}{2}, x_3 + x_4, 0\right)$
$\therefore$(1, 1, 0) = $\left(\frac{x_1+x_2}{2}, x_3 + x_4, 0\right)$
Thus we get, $\frac{x_1+x_2}{2} = 1$, $x_3 + x_4 = 1$
Let us take $x_1 = 2, x_2 = 0, x_3 = 1, x_4 = 0, x_5 = 0$.

Then we get one particular solution of T(u) = (1, 1, 0) is u_0 = (2, 0, 1, 0, 0)
So the complete solution of the equation T(u) = (1, 1, 0)
is the linear variety (2, 0, 1, 0, 0) + N(T)

i.e. The set (2, 0, 1, 0, 0) + $\{(a, -a, b, -b, c)/a, b, c \in R\}$,
Which is same as $\{(a+2), -a, (b+1), -b, c)/a, b, c \in R\}$
In other words: The T-pre-image (1, 1, 0) of is this linear variety.

Example-2 Let T: $R^3 \rightarrow R^2$ be a linear transformation defined by $T(x_1, x_2, x_3) = (x_1 + x_2, 2x_2 + x_3)$ then solve the equation T(u) = (2, 4), u$\in R^3$.
Solution: Frist calculate the value of T(u) i.e. by $T(x_1, x_2, x_3) = (x_1 + x_2, 2x_2 + x_3)$
The associated homogeneous equation leads to the equations
i.e. $T(x_1, x_2, x_3) = 0$

$$x_1 + x_2 = 0, 2x_2 + x_3 = 0.$$

Solving these, we get $x_1 = -x_2$, $x_3 = -2x_2$
Thus, the kernel of T is the set of all vectors of form $(-x_2, x_2, -2x_2)$,
If we take $-x_2 = a \forall a \in R$
N(T) = $[(\mathrm{a}, -\mathrm{a}, 2\mathrm{a})]$
$\therefore$ N(T) = $[(1, -1, 2)]$

Now find T(u) = (2, 4), u$\in R^3$.
Since T(u) = $T(x_1, x_2, x_3) = (x_1 + x_2, 2x_2 + x_3)$
$\therefore$ (2, 4) = $(x_1 + x_2, 2x_2 + x_3)$
Thus we get, $x_1 + x_2 = 2$, $2x_2 + x_3 = 4$ $\Rightarrow x_1 = 2 - x_2$, $x_3 = 4 - 2x_2$
Let us take $x_2 = -1$ then we get $x_1 = 3, x_3 = 6$
Then we get one particular solution of T(u) = (2, 4) is u_0 = (3, -1, 6)
So the complete solution of the equation T(u) = (2, 4)
is the linear variety (3, -1, 6) + N(T)
i.e. The set (3, -1, 6)+ $\{(a, -a, 2a)/a \in R\}$,
Which is same as $\{(a+3), -(a+1), (2a+6)/a \in R\}$
In other words: The T-pre-image (2, 4) of is this linear variety.

Example-3 Let T: $R^4 \rightarrow R^3$ be a linear transformation defined by $T(x_1, x_2, x_3, x_4) = (x_1 - x_4, x_2 + x_3, x_3 - x_4)$ then solve the equation T(u) = (1,2, 3), u$\in R^4$.
Solution: Frist calculate the value of T(u)
i.e. by $T(x_1, x_2, x_3, x_4) = (x_1 - x_4, x_2 + x_3, x_3 - x_4)$
The associated homogeneous equation leads to the equations
i.e. $T(x_1, x_2, x_3, x_4) = 0$

$$x_1 - x_4 = 0, x_2 + x_3 = 0 \;\; x_3 - x_4 = 0.$$

Solving these, we get $x_1 = x_4$, $x_2 = -x_3$, $x_3 = x_4$
Thus, the kernel of T is the set of all vectors of form $(x_4, -x_4, x_4, x_4)$
If we take $x_4 = a \forall a \in R$
N(T) = $[(\mathrm{a}, -\mathrm{a}, \mathrm{a}, \mathrm{a})]$ i.e. N(T) = $[a(1, -1, 1, 1)]$
$\therefore$ N(T) = $[(1, -1, 1, 1)]$

Now find T(u) = (1,2, 3), u$\in R^4$
Since T(u) = $T(x_1, x_2, x_3, x_4) = (x_1 - x_4, x_2 + x_3, x_3 - x_4)$
$\therefore$ (1,2, 3) = $(x_1 - x_4, x_2 + x_3, x_3 - x_4)$

Thus we get, $x_1 - x_4 = 1$, $x_2 + x_3 = 2, x_3 - x_4 = 3$ $\Rightarrow x_1 = 1 + x_4$, $x_2 = 2 - x_3$, $x_3 = 3 + x_4$
$\therefore x_2 = 2 - 3 - x_4 = -1 - x_4$
Let us take $x_4 = 1$ then we get $x_1 = 2, x_2 = -2, x_3 = 4$
Then we get one particular solution of T(u) = (1,2, 3)is u_0 = (2, -2, 4,1)
So the complete solution of the equation T(u) = (1,2, 3)
is the linear variety (2, -2, 4,1)+ N(T)
i.e. The set (2, -2, 4,1)+ $\{(a, -a, a, a)/a \in R\}$
Which is same as $\{(a + 2), -(a + 2), (a + 4), (a + 1)/a \in R\}$
In other words: The T-pre-image (2, 4) of is this linear variety.
OR
Let us take $x_4 = 0$ then we get $x_1 = 1, x_2 = -1, x_3 = 3$
Then we get one particular solution of T(u) = (1,2, 3)is u_0 = (1, -1, 3,0)

Example-4 Let T: $R^4 \rightarrow R^3$ be a linear transformation defined by$T(e_1) = f_1$, $T(e_2) = f_2$, $T(e_3) = f_1 + f_2$and $T(e_4) = -f_2 - f_3$. Where $\{e_1, e_2, e_3, e_4\}$ is the standard basis for R^4 and $\{f_1, f_2, f_3\}$ is the standard basis for R^3 then solve the equation T(u) = (1, 2, 3).
Solution: Frist calculate the value of T(u)
i.e. $T(x_1, x_2, x_3, x_4)$ Here u= $(x_1, x_2, x_3, x_4) \in R^4$.
Since T is a linear map
$\therefore\ T(x_1, x_2, x_3, x_4) = x_1 T(e_1) + x_2 T(e_2) + x_3 T(e_3) + x_4 T(e_4)$
$= x_1 f_1 + x_2 f_2 + x_3(f_1 + f_2) + x_4(-f_2 - f_3)$ (Put the given values)
$= (x_1 + x_3)f_1 + (x_2 + x_3 - x_4)f_2 + (-x_4)f_3$
$= \left((x_1 + x_3), (x_2 + x_3 - x_4), (-x_4)\right).(f_1, f_2, f_3)$
$T(x_1, x_2, x_3, x_4) = \left((x_1 + x_3), (x_2 + x_3 - x_4), (-x_4)\right)$
The associated homogeneous equation leads to the equations
i.e. $T(x_1, x_2, x_3, x_4) = 0$
$$x_1 + x_3 = 0, x_2 + x_3 - x_4 = 0, -x_4 = 0.$$
Solving these, we get $x_1 = -x_3$, $x_2 = -x_3$, $x_4 = 0$
Thus, the kernel of T is the set of all vectors of form$(-x_3, -x_3, x_3, 0)$,
If we take $x_3 = a \forall a \in R$
N(T) = $[(-a, -a, a, 0)]$ i.e. N(T) = $[a(-1, -1,1,0)]$
$\therefore$ N(T) = $[(-1, -1,1,0)]$
Now find T(u) = (1, 2, 3).
Since T(u) = $T(x_1, x_2, x_3, x_4) = \left((x_1 + x_3), (x_2 + x_3 - x_4), (-x_4)\right)$
$\therefore$((1, 2, 3).= $\left((x_1 + x_3), (x_2 + x_3 - x_4), (-x_4)\right)$
Thus we get, $x_1 + x_3 = 1$, $x_2 + x_3 - x_4 = 2, -x_4 = 3$
Let us take $x_1 = 1 - x_3, x_2 = -1 - x_3, x_4 = -3$.
If we take $x_3 = 0$,then $x_1 = 1, x_2 = -1, x_4 = -3$
Then we get one particular solution of T(u) = (1, 2, 3) is u_0 = (1, -1, 0, -3)
So the complete solution of the equation T T(u) = (1, 2, 3)
is the linear variety (1, -1, 0, -3)+ N(T)
i.e. The set (1, -1, 0, -3)+ $\{(-a, -a, a, 0)/a \in R\}$,
Which is same as $\{(-a + 1, -a - 1, a, -3)/a \in R\}$
In other words: The T-pre-image (1, 2, 3) of is this linear variety.

Theorem:-(dual basis existence theorem) Let V be an n-dimensional vector space and let B = { $x_1, x_2, ... x_n$}be a basis of V. Then prove that there is a uniquely determined basis B* = {$f_1, f_2, f_3, ..., f_n$}of V* such that $f_i(x_i) = \delta_{ij}$ i,j = 1, 2, 3, …,n.

Proof: B = { $x_1, x_2, ... x_n$}be a basis of V and (1,0,0,0,---,0) is an ordered set of n scalars, then there exists a unique linear functional f_1 on V such that
$f_1(x_1) = 1, f_2(x_2) = 0, f_3(x_3) = 0, \ldots, f_n(x_n) = 0$.
In fact
For each i = 1, 2, 3,…,n there exists a unique linear functional f_i on V such that
$f_i(x_i) = \delta_{ij}$ i,j = 1, 2, 3, …,n.
Let B* = {$f_1, f_2, f_3, ..., f_n$}
We shall show that B* is a basis of V*.
For this, first we show that B* is linearly independent.
Let $\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 + \ldots + \alpha_n f_n = 0 \ \forall \alpha_i \epsilon R, i = 1,2,3, \ldots, n$
$(\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 + \ldots + \alpha_n f_n)(x) = 0(x) \ \forall x \epsilon V$
$\alpha_1 f_1(x) + \alpha_2 f_2(x) + \alpha_3 f_3(x) + \ldots + \alpha_n f_n)(x) = 0$
i.e. $\sum_{i=1}^{n} \alpha_i f_i(x_j) = 0, \ \forall j = 1,2,3, \ldots, n$
$\Rightarrow \sum_{i=1}^{n} \alpha_i \delta_{ij} = 0$
$\Rightarrow \alpha_i = 0, = 1,2,3, \ldots, n$
Hence B* is linearly independent.

Definition:- Annihilators :
Let W be a subset of a vector space V over a field k and V* its dual. Let W be a subset of V which is not necessarily a subspace. Then a linear functional $f \epsilon V^*$ is called an annihilator of W if $f(x) = 0$ for every x϶W.
It is denoted by W^0.
i.e. The set of all linear functional f on V such that $f(x) = 0$ for every x϶V.
i.e. $f(w) = 0$, is called an annihilator of W.
Also $W^0 = \{f \epsilon V^* : f(x) = 0 \forall x \in V\}$
Note:- Annihilator of V is the zero functional on V. and $\{0\}^0 = V^*$.

Theorem:- Let W be a non-empty subset of a vector space V. Then prove that the annihilator $W^0$of W is subspace of V^*. **(OR)** Prove that the W^0 is subspace of V^*.
Proof:- By the definition of W^0,
It is clear that $0 \in W^0$ and $W^0 \subseteq V^*$.
Now suppose $\phi_1, \phi_2 \in W^0$ and for any scalars $a, b \in R$ and for any $\alpha \in W$,

$$(a\phi_1+b\phi_2)(\alpha) = a\phi_1(\alpha)+b\phi_2(\alpha)$$
$$= a.0 + b.0 \quad (\because \phi_1, \phi_2 \in W^0)$$
$$= 0$$

$\therefore a\phi_1+b\phi_2 \in W^0$

Hence, W^0 is subspace of V^*.

Note:- W^0 is subspace of V^*, whether W is a subspace of V or not.

Theorem:- Let V be a finite dimensional vector space over the field F and let W be a subspace of V. Then Prove that dim W + dim W^0 = dim V.

(OR) If W is an m-dimensional subspace of an n-dimensional vector space V. then show that the annihilator W^0 is an (n-m) dimensional subspace of V^*.

Proof:-

Let V be a finite dimensional vector space over the field F.

Let dimW = m

Let W be a subspace of V. Then W^0 is subspace of V^*.

Since W is a subspace of V so that

dimW ≤dim dimW

i.e. m≤n.

Let { $x_1, x_2,...x_m$}be a basis of W.

So it can be extended to form a basis of V.

Choose vectors $x_{m+1}, x_{m+2},... x_n$ in V such that B = { $x_1, x_2,...x_m , x_{m+1}, x_{m+2},... x_n$ }is basis of V.

Let {$f_1, f_2, f_3, ...,f_n$}be basis of V^* which is the dual to B.

Now we claim that { $f_{m+1}, f_{m+2},... f_n$ } is basis of W^0.

Obviously, $f_i \in W^0, \ \forall i \geq m+1$ because $f_i(x_j)\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

And $\delta_{ij} = 0$ if $i \geq m+1$ and j≤m.

Since { $f_{m+1}, f_{m+2},... f_n$ } is a subset of linearly independent

Now we show that { $f_{m+1}, f_{m+2},... f_n$ } spans W^0.

Let $f \in W^0$ be an arbitrary linear functional,

So that $f(x_i) = 0$ for 1 ≤ i ≤ m __________(1)

$W^0 \subseteq V^*$ and $f \in V^*$

But {$f_1, f_2, f_3, ...,f_n$} generates V^*.

$$\therefore f = \sum_{i=1}^{n} f(x_i)f_i$$

$$= f(x_1)f_1 + f(x_2)f_2 + \cdots + f(x_m)f_m + f(x_{m+1})f_{m+1} + f(x_{m+2})f_{m+2} + \cdots + f(x_n)f_n$$

$$= f(x_{m+1})f_{m+1} + f(x_{m+2})f_{m+2} + \cdots + f(x_n)f_n = \sum_{i=m+1}^{n} f(x_i)f_i$$

This shows that { $f_{m+1}, f_{m+2},... f_n$ }} spans W^0.

Thus, { $f_{m+1}, f_{m+2},... f_n$ } is basis of W^0.

Accordingly,

dim W^0 = n – m = dim V - dim W.

Corollary:-If W and W_1 are two subspaces of a vector space V which are annihilated by the subspace W^0then dimW = dimW_1.

Proof:- W and W_1 are two both annihilated by the subspace $W^0$and both are subspaces of V then we have

dim W + dim W^0 = dim V ________________(1)

dim W_1 + dim W^0 = dim V ________________(2)

Now subtract (2) from (1) then we get

dim W = dim W_1

Theorem:- If W and W_1 are two subspaces of a finite dimensional vector space V, then $W_1 = W_2$ if and only if $W_1^0 = W_2^0$.

Proof:- If $W_1 = W_2$ then obviously $W_1^0 = W_2^0$.

Let us suppose that $W_1 \neq W_2$

Then there is at least one vector W_1in which not in W_2.

Suppose $x \in W_2$ and $x \notin W_1$

Then there a linear functional *f* such that *f*(y) = 0 $y \in W$ but *f*(x) $\neq 0$

This implies that $f \in W_1^0$,but $f \notin W_2^0$ and thus $W_1^0 \neq W_2^0$.

Hence $W_1^0 = W_2^0$ if $W_1 = W_2$

Bilinear forms

Definition:- Bilinear form (or) 2-Form: (or) bilinear functional

Suppose VC is finite dimensional vector space over a field R. Let $u_1, u_2, v_1, v_2 \in V$ and $a, b \in R$ be arbitrary. A mapping $T: V \times V \to R$ is a bilinear (or bilinear functional) on V. if following are satisfied:

(i) T(u, $av_1 + bv_2) = a.T(u, v_1) + b.T(u, v_2)$

(ii) T(u, $av_1 + bv_2) = a.T(u, v_1) + b.T(u, v_2)$

Note:-

→We express condition (i) by saying *f* is linear in its first variable (co-ordinate) and condition (ii) by saying *f* is linear in its second variable (co-ordinate).

→ Such mapping *f* id also known as Sesqui-linear form.

Example:

Prove that the zero function from $T: V \times V \to R$ is a bilinear on V.

i.e. Let from $T: V \times V \to R$ defined by from $T(u, v,) = 0,\ \forall u, v \in V$ is bilinear on V.

Solution:

Let $u_1, u_2, v_1, v_2 \in V$ and $a, b \in R$

$\therefore T(u_1, v) = T(u_2, v) = T(u, v_1) = T(u, v_2) = 0$

Since T(u, $av_1 + bv_2) = 0$

$$= a.0 + b.0$$
$$= a.T(u, v_1) + b.T(u, v_2)$$

Similarly,

$$\mathrm{T}(au_1 + bu_2, v) = 0$$
$$= a.0 + b.0$$
$$= a.T(u_1, v) + b.T(u_2, v)$$

$\therefore T$ is bilinear form.

Example: Let V= R^3. Suppose u =(x_1,x_2,x_3) and v = (y_1,y_2,y_3) $\in R^3$ and defined by $f(u,v) = x_1y_2 - 3x_2y_3 + x_3y_1$ then Show that f is a bilinear for.

Solution:-

Let $u = (x_1x_2, x_3)$, $v = (y_1y_2, y_3)$and $w = (z_1z_2, z_3) \in R^3$ and $a, b \in R$

$\therefore au + bw = (ax_1 + bz_1, ax_2 + bz_2, ax_3 + bz_3)$

Now,

$$\begin{aligned} f(au + bw, v) &= (ax_1 + bz_1)y_2 - 3(ax_2 + bz_2)y_3 + (ax_3 + bz_3)y_1 \\ &= a(x_1y_2 - 3x_2y_3 + x_3y_1) + b(z_1y_2 - 3z_2y_3 + z_3y_1) \\ &= af(u,v) + bf(w,v) \end{aligned}$$

Similarly,

$$\begin{aligned} f(u, av + bw) &= x_2(ay_2 + bz_2) - 3x_2(ay_3 + bz_3) + x_3(ay_1 + bz_1) \\ &= a(x_1y_2 - 3x_2y_3 + x_3y_1) + b(x_1z_2 - 3x_2z_3 + x_3z_1) \\ &= af(u,v) + bf(u,w) \end{aligned}$$

$\therefore f$ is bilinear form.

Example:

Which of the following functions *f* define on vectors *u* = (x_1, x_2) and *v* = (y_1, y_2) in R^2 are bilinear form?

(1) $f(u, v) = x_1y_2 - x_2y_1$

(2) $f(u, v) = (x_1 - y_1)^2 + x_2y_2$.

Solution:-(1)

Let $u = (x_1x_2)$, $v = (y_1y_2)$and $w = (z_1z_2) \in R^2$ and $a, b \in R$

$\therefore au + bw = (ax_1 + bz_1, ax_2 + bz_2, ax_3 + bz_3)$

And $ay + bw = (ay_1 + bz_1, ay_2 + bz_2, ay_3 + bz_3)$

Now,

$$\begin{aligned} f(au + bw, v) &= (ax_1 + bz_1)y_2 - (ax_2 + bz_2)y_1 \\ &= a(x_1y_2 - x_2y_1) + b(z_1y_2 - z_2y_1) \\ &= af(u,v) + bf(w,v) \end{aligned}$$

Similarly,

$$\begin{aligned} f(u, av + bw) &= x_1(ay_2 + bz_2) - x_2(ay_1 + bz_1) \\ &= a(x_1y_2 - x_2y_1) + b(x_1z_2 - x_2z_1) \\ &= af(u,v) + bf(u,w) \end{aligned}$$

$\therefore f$ is bilinear form.

Solution:-(2)

Let $u = (x_1x_2)$, $v = (y_1y_2)$and $w = (z_1z_2) \in R^2$ and $a, b \in R$

$\therefore au + bw = (ax_1 + bz_1, ax_2 + bz_2, ax_3 + bz_3)$

And $ay + bw = (ay_1 + bz_1, ay_2 + bz_2, ay_3 + bz_3)$

Now,

$$\begin{aligned} f(au, v) &= (ax_1 - y_1)^2 + ax_2y_2 \\ &= a^2{x_1}^2 - 2ax_1y_1 + {y_1}^2 + ax_2y_2 \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(i)} \end{aligned}$$

And

$af(u,v) = a[(x_1 - y_1)^2 + x_2y_2]$

$$= ax_1{}^2 - 2ax_1y_1 + ay_1{}^2 + ax_2y_2 \underline{\hspace{6cm}} \text{(ii)}$$

From (i) and (ii)

$$f(au, v) \neq af(u, v)$$

$\therefore f$ is not a bilinear form on R^2.

Example:

Let ϕ and ψ be linear functional on a vector space V over R. Define a map $T: V \times V \rightarrow R$ by the formula $T(u, v) = \phi(u).\psi(v)\ \forall u, v \in V$. Then show that T is bilinear on V.

Solution:

Let u,v,w$\in V$ and $a, b \in R$

$$\begin{aligned}
\therefore T(au + bw, v) &= \phi(au + bw).\psi(v) \\
&= [a\phi(u) + b\phi(w)].\psi(v) \quad (\because \phi \text{ is linear map.}) \\
&= a\phi(u).\psi(v) + b\phi(w).\psi(v) \\
&= aT(u, v) + bT(w, v)
\end{aligned}$$

Similarly,

$$\begin{aligned}
T(u, av + bw) &= \phi(u).\psi(av + bw) \\
&= \phi(u).[a\psi(v) + b\psi(w)] \quad (\because \phi \text{ is linear map.}) \\
&= a\phi(u).\psi(v) + b\phi(u).\psi(w) \\
&= aT(u, v) + bT(u, w)
\end{aligned}$$

$\therefore T$ is bilinear form.

Example:

Define a map $T: R^n \rightarrow R$ by the formula $T(u, v) = \sum_{i=1}^{n} a_i b_i$ where $u = (a_1, a_2, \dots, a_n)$ and $v = (b_1, b_2, \dots, b_n)$. Then show that T is bilinear on R^n.

Solution:

Let u,v,w$\in R^n$ where $u = (a_1, a_2, \dots, a_n)$ and $v = (b_1, b_2, \dots, b_n)$ and $w = (c_1, c_2, \dots, c_n)$ and $\alpha, \beta \in R$

$$\begin{aligned}
\therefore T(\alpha u + \beta w, v) &= \sum_{i=1}^{n} (\alpha a_i + \beta c_i) b_i \\
&= \textstyle\sum_{i=1}^{n} (\alpha a_i b_i + \beta c_i b_i) \\
&= \alpha \textstyle\sum_{i=1}^{n} a_i b_i + \beta \textstyle\sum_{i=1}^{n} c_i b_i \\
&= aT(u, v) + bT(w, v)
\end{aligned}$$

Similarly,

$$\begin{aligned}
T(u, av + bw) &= \sum_{i=1}^{n} a_i(\alpha b_i + \beta c_i) \\
&= \textstyle\sum_{i=1}^{n} (\alpha a_i b_i + \beta a_i c_i) \\
&= \alpha \textstyle\sum_{i=1}^{n} a_i . b_i + \beta \textstyle\sum_{i=1}^{n} a_i c_i \\
&= aT(u, v) + bT(u, w)
\end{aligned}$$

$\therefore T$ is bilinear form.

Note :- The set of all bilinear forms on V denoted by B(V).