Unit-02 BSc. Sem-V MAT-304

2, 3, 4.

§ *Solve the given L.P.Problem by simplex method*

(1) Maximize $z = 7x_1 + 5x_2$

subject to,

$$x_1 + 2x_2 \le 6$$
$$4x_1 + 3x_2 \le 12$$
$$\text{and } x_1, x_2 \ge 0.$$

ROSHAN K. PATEL
→8140640791

⇒

Here, $\max(z) = -\min(-z)$

$\therefore \min(-z) = -7x_1 - 5x_2 + 0x_3 + 0x_4$

subject to,

$$x_1 + 2x_2 + x_3 + 0x_4 = 6$$
$$4x_1 + 3x_2 + 0x_3 + x_4 = 12$$
$$\text{and } x_i \ge 0 ; i = 1, 2, 3, 4.$$

Now, $n = 4$, $m = 2 \Rightarrow n - m = 4 - 2 = 2$

Here, x_1 and x_2 are non-basic variables i.e $x_1 = x_2 = 0$ and $x_3 = 6$ and $x_4 = 12$ are basic variables.

Which is the initial basic feasible solution.

The initial simplex table is presented as follows.

Table : - 1

C_B	Vari.in basis	c_j → X_B	-7 x_1	-5 x_2	0 x_3	0 x_4	R.Ratio X_B / x_1
0	x_3	6	1	2	1	0	$6/1 = 6$
0	x_4 ←	12	4	3	0	1	$12/4 = 3$
$-z = \sum C_B X_B = 0$			-7	-5	0	0	$c_j - z_j$

↑ most –ve

Here, the most negative $c_j - z_j$ is '-7' and it occur in the column of x_1.

∴ We take the replacement ratio X_B / x_1.

Now, the minimum positive ratio is '3' and it occur in the row of x_4.

∴ x_4 will leave the basis and x_1 will enter in the basis. Also '4' is the key element.

∴ Divide all the elements of the x_4 row by '4' and convert all the elements of column x_1 to zero using unity.

Table : - 2

C_B	Vari.in basis	c_j	-7	-5	0	0	R.Ratio
		X_B	x_1	x_2	x_3	x_4	
0	x_3	3	0	$5/4$	1	$-1/4$	
-7	x_1	3	1	$3/4$	0	$1/4$	
$-z = \sum C_{B_i} X_{B_i} = -21$			0	$1/4$	0	$7/4$	$c_j - z_j$

Here, all $c_j - z_j \geq 0$

$\therefore$ This the optimum solution with $x_1 = 3$, $x_2 = 0$, $x_3 = 3$ and $x_4 = 0$.

$$\min(-z) = -7(3) - 5(0) = -21$$

$$\therefore \max(z) = -\min(-z) = 21$$

(2) Maximize $z = 3x_1 + 2x_2$

subject to,

$$-x_1 + 2x_2 \leq 4$$
$$3x_1 + 2x_2 \leq 14$$
$$-x_1 + x_2 \geq -3$$
$$\text{and } x_1, x_2 \geq 0.$$

$\Rightarrow$

Here, $\max(z) = -\min(-z)$

$\therefore \min(-z) = -3x_1 - 2x_2$

subject to,

$$-x_1 + 2x_2 \leq 4$$
$$3x_1 + 2x_2 \leq 14$$
$$x_1 - x_2 \leq 3$$
$$\text{and } x_1, x_2 \geq 0.$$

Now we add slack variables in the constraints.

$\therefore \min(-z) = -3x_1 - 2x_2 + 0x_3 + 0x_4 + 0x_5$

subject to,

$$-x_1 + 2x_2 + x_3 + 0x_4 + 0x_5 = 4$$
$$3x_1 + 2x_2 + 0x_3 + x_4 + 0x_5 = 14$$
$$x_1 - x_2 + 0x_3 + 0x_4 + x_5 = 3$$
$$\text{and } x_i \geq 0;\ i = 1, 2, \ldots 5.$$

$n = 5$, $m = 3 \Rightarrow n - m = 5 - 3 = 2$

Here, x_1 and x_2 are non-basic variables i.e. $x_1 = x_2 = 0$ and $x_3 = 4$, $x_4 = 14$ and $x_5 = 3$ are basic variables.

Which is the initial basic feasible solution.

The initial simplex table is presented as follows.

Table: - 1

C_B	Vari. in basis	c_j	– 3	– 2	0	0	0	R.Ratio
		X_B	x_1	x_2	x_3	x_4	x_5	X_B / x_1
0	x_3	4	– 1	2	1	0	0	4 / (–1) = – 4
0	x_4	14	3	2	0	1	0	14 / 3
0	x_5	3	1	– 1	0	0	1	3 / 1 = 3
$-z = \sum C_{B_i} X_{B_i} = 0$			– 3	– 2	0	0	0	c_j - z_j

most –ve

Here, the most negative $c_j - z_j$ is '– 3' and it occur in the column of x_1.

∴ We take the replacement ratio X_B / x_1.

Now, the minimum positive ratio is '3' and it occur in the row of x_5.

∴ x_5 will leave the basis and x_1 will enter in the basis. Also '1' is the key element.

∴ Convert all the elements of column x_1 to zero using unity.

Table : - 2

C_B	Vari. in basis	c_j	– 3	– 2	0	0	0	R.Ratio
		X_B	x_1	x_2	x_3	x_4	x_5	X_B / x_2
0	x_3	7	0	1	1	0	1	7 / 1 = 7
0	x_4	5	0	5	0	1	– 3	5 / 5 = 1
– 3	x_1	3	1	– 1	0	0	1	3 / (– 1) = 3 –3
$-z = \sum C_B X_B = -21$			0	– 5	0	0	3	c_j - z_j

most –ve

Here, all $c_j - z_j$ are not non-negative. The most negative $c_j - z_j$ is appears in the column of x_2 and it is '– 5'.

$\therefore$ Replacement ratio = X_B / x_2 .

The minimum positive ratio is ' 1 ' and it occur in the row of x_4.

$\therefore$ x_4 will leave the basis and x_2 will enter in the basis. Also '5' is the key element.

$\therefore$ Divide all the elements of the x_4 row by '5' and convert all the elements of column x_2 to zero using unity.

Table : - 3

C_B	Vari. in basis	c_j / X_B	-3 / x_1	-2 / x_2	0 / x_3	0 / x_4	0 / x_5	R.Ratio
0	x_3	6	0	0	1	$-1/5$	$8/5$	
-2	x_2	1	0	1	0	$1/5$	$-3/5$	
-3	x_1	4	1	0	0	$1/5$	$2/5$	
$-z = \sum C_{B_i} X_{B_i} = -14$			0	0	0	1	0	$c_j - z_j$

Here, all $c_j - z_j \ge 0$

$\therefore$ This the optimum solution with $x_1 = 4$, $x_2 = 1$, $x_3 = 6$ $x_4 = 0$ and $x_5 = 0$.

$\therefore$ max (z) = 3 (4) + 2 (1)

= 14.

(3) **Minimize $z = x_1 - 3x_2 + 2x_3$**

subject to,

$$3x_1 - x_2 + 3x_3 \le 7$$
$$-2x_1 + 4x_2 \le 12$$
$$-4x_1 + 3x_2 + 8x_3 \le 10$$
and $x_1, x_2, x_3 \ge 0.$

2006 $-x_1 + 4x_2 \le 12$

Ans -- (-9)

$\Rightarrow$

Here, min(z) = $x_1 - 3x_2 + 2x_3 + 0x_4 + 0x_5 + 0x_6$

subject to,

$$3x_1 - x_2 + 3x_3 + x_4 + 0x_5 + 0x_6 \le 7$$
$$-2x_1 + 4x_2 + 0x_4 + x_5 + 0x_6 \le 12$$
$$-4x_1 + 3x_2 + 8x_3 + 0x_4 + 0x_5 + x_6 \le 10$$
and $x_i \ge 0$; i = 1, 2, ...6.

Now, n = 6, m = 3 $\Rightarrow$ n − m = 6 − 3 = 3

Here, x_1, x_2 and x_3 are non-basic variables and $x_4 = 7$, $x_5 = 12$ and $x_6 = 10$ are basic variables. This is the initial basic feasible solution.

The initial simplex table is presented as follows.

Table: - 1

C_B	Vari. in basis	c_j / X_B	1 / x_1	-3 / x_2	2 / x_3	0 / x_4	0 / x_5	0 / x_6	R.Ratio X_B / x_2
0	x_4	7	3	-1	3	1	0	0	$7/(-1) = -7$
0	x_5	12	-2	4	0	0	1	0	$12/4 = 3$
0	x_6	10	-4	3	8	0	0	1	$10/3$
$z = \sum C_{B_i} X_{B_i} = 0$			1	-3	2	0	0	0	$c_j - z_j$

most –ve

Here, the most negative $c_j - z_j$ is '-3' and it occur in the column of x_2.

$\therefore$ The replacement ratio $= X_B / x_2$.

Now, the minimum positive ratio is '3' and it occur in the row of x_5.

$\therefore$ x_5 will leave the basis and x_2 will enter in the basis. Also '4' is the key element.

$\therefore$ Divide all the elements of the x_5 row by '4' and convert all the elements of column x_2 to zero using unity.

Table: - 2

C_B	Vari. in basis	c_j / X_B	1 / x_1	-3 / x_2	2 / x_3	0 / x_4	0 / x_5	0 / x_6	R.Ratio X_B / x_1
0	x_4	10	$\frac{5}{2}$	0	3	1	$\frac{1}{4}$	0	$10/(5/2) = 4$
-3	x_2	3	$-\frac{1}{2}$	1	0	0	$\frac{1}{4}$	0	$3/(-1/2) = -6$
0	x_6	1	$-\frac{5}{2}$	0	8	0	$-\frac{3}{4}$	1	$1/(-5/2) = -2/5$
$z = \sum C_B X_B = -9$			$-\frac{1}{2}$	0	2	0	$\frac{3}{4}$	0	$c_j - z_j$

Here, all $c_j - z_j$ are not non-negative. The most negative $c_j - z_j$ is appears in the column of x_2 and it is ' $-\frac{1}{2}$ '.

$\therefore$ Replacement ratio = $\frac{X_B}{x_1}$.

The minimum positive ratio is ' 4 ' and it occur in the row of x_4.

$\therefore$ x_4 will leave the basis and x_2 will enter in the basis. Also ' $\frac{5}{2}$ ' is the leading element.

$\therefore$ Divide all the elements of the x_4 row by ' $\frac{5}{2}$ ' and convert all the elements of column x_1 to zero using unity.

ble: - 3

C_B	Vari. in basis	c_j / X_B	1 / x_1	-3 / x_2	2 / x_3	0 / x_4	0 / x_5	0 / x_6	R.Ratio
1	x_1	4	1	0	$\frac{6}{5}$	$\frac{2}{5}$	$\frac{1}{10}$	0	
-3	x_2	5	0	1	$\frac{3}{5}$	$\frac{1}{5}$	$\frac{3}{10}$	0	
0	x_6	11	0	0	11	1	$-\frac{1}{2}$	1	
$z = \sum C_{B_i} X_{B_i} = -11$			0	0	$\frac{13}{5}$	$\frac{1}{5}$	$\frac{4}{5}$	0	$c_j - z_j$

Here, all $c_j - z_j \geq 0$

$\therefore$ We obtained the optimum solution with $x_1 = 4$, $x_2 = 5$, $x_3 = 0$, $x_4 = 0$, $x_5 = 0$ and $x_6 = 11$.

$\therefore$ min (z) = 1 (4) – 3 (5)+ 2 (0)

= – 11.

§ *Two phase method for solving L.P. Problem :-*

Phase : - 1

Step-1 In minimization case.

Assign a cost '1' to each artificial variable and a cost '0' to all other variables (including slack and surplus variables) in the auxiliary equation i.e. new objective function. g(x)

$$\text{We write } g(x) = \sum_{i=1}^{p} A_i \text{ where } x_i \text{ are artificial variables.}$$

Step-2

Write down the auxiliary L.P.Problem in which new objective function g(x) is to be minimize subject to the given constraints.

Step-3

Use Simplex method to solve the auxiliary L.P.Problem.

⇒ If min $g(x) > 0$ and if at_least one artificial variable appears in the optimal basis at a ' +ve ' level then the given L.P.Problem does not possess any feasible solution.

⇒ If min $g(x) = 0$ and if at_least one artificial variable appears in the optimal basis at zero level then we proceed to Phase-2.

⇒ If min $g(x) = 0$ and if no artificial variable appears in the optimal basis then we proceed to Phase-2.

Phase : - 2

Step-1

Assign a actual cost to the variables in the original objective function and a cost '0' to every artificial variable that appears in the basis at the zero level. i.e. remove the artificial variables and the column of the artificial variables in Phase-2 which are non-basic at the end of Phase-1.

The original objective function is to be minimize subject to the given constraints by applying Simplex method.

Artificial variables : -

If the first (initial) basic feasible solution for the given L.P. Problem is not available then the new variables are added to act as a basic variable in the constraints which are called artificial variables.

Artificial variables have no meaning to the original problem. They are nearly added so that we get a initial basic feasible solution and we can start the simplex method.

§ ***Solve the given L.P.Problem by Two phase method***

(1) **Minimize $z = 4x_1 + 5x_2$**

subject to,

$$2x_1 + x_2 \le 6$$
$$x_1 + 2x_2 \le 5$$
$$x_1 + x_2 \ge 1$$
$$x_1 + 4x_2 \ge 2 \text{ and } x_1, x_2 \ge 0.$$

$\Rightarrow$ Here,

Minimize $z = 4x_1 + 5x_2$

subject to,

$$2x_1 + x_2 + x_3 = 6$$
$$x_1 + 2x_2 + x_4 = 5$$
$$x_1 + x_2 - x_5 = 1$$
$$x_1 + 4x_2 - x_6 = 2$$
$$\text{and } x_i \ge 0;\ i = 1, 2, \ldots..6.$$

Now, $n = 6$, $m = 4$ $\Rightarrow$ $n - m = 6 - 4 = 2$

x_1, x_2 are non-basic variables

$\therefore$ $x_3 = 6$, $x_4 = 5$, $x_5 = -1$ and $x_6 = -2$

As $x_5 < 0$, $x_6 < 0$ This solution is not feasible.

$\therefore$ We shall add variables x_7, x_8 known as artificial variables.

$\therefore$ $x_1 + x_2 - x_5 + x_7 = 1$

$x_1 + 4x_2 - x_6 + x_8 = 2$

Now, $n = 8$, $m = 4$ $\Rightarrow$ $n - m = 8 - 4 = 4$

x_1, x_2, x_5, x_6 are non-basic variables and basic variables $x_3 = 6$, $x_4 = 5$, $x_7 = 1$ and $x_8 = 2$. Which is the initial basic feasible solution.

We solve this L.P.Problem by two phase simplex method.

Phase-1 Here, objective function $g(x) = x_7 + x_8$

The initial simplex table is presented as follows.

Table: - 1

C_B	Vari.in basis	c_j / X_B	0 x_1	0 x_2	0 x_3	0 x_4	0 x_5	0 x_6	1 x_7	1 x_8	R.Ratio X_B / x_2
0	x_3	6	2	1	1	0	0	0	0	0	6 / 1 = 6
0	x_4	5	1	2	0	1	0	0	0	0	5 / 2
1	x_7	1	1	1	0	0	– 1	0	1	0	1 / 1 = 1
1	x_8	2	1	4	0	0	0	– 1	0	1	1 / 2 ←
$g(x) = 3$			– 2	– 5 ↑	0	0	1	1	0	0	$c_j - z_j$

most -ve

'**4**' is the key element.

Table: - 2

C_B	Vari.in basis	c_j / X_B	0 x_1	0 x_2	0 x_3	0 x_4	0 x_5	0 x_6	1 x_7	1 x_8	R.Ratio X_B / x_1
0	x_3	$\frac{11}{2}$	$\frac{7}{4}$	0	1	0	0	$\frac{1}{4}$	0	$-\frac{1}{4}$	22 / 7
0	x_4	4	$\frac{1}{2}$	0	0	1	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	4 / (½) = 8
1	x_7	$\frac{1}{2}$	$\frac{3}{4}$	0	0	0	– 1	$\frac{1}{4}$	1	$-\frac{1}{4}$	(½)/(¾) = 2 / 3
0	x_2	$\frac{1}{2}$	$\frac{1}{4}$	1	0	0	0	$-\frac{1}{4}$	0	$\frac{1}{4}$	(½)/(¼) = 2
$g(x) = \frac{1}{2}$			$-\frac{3}{4}$ ↑	0	0	0	1	$-\frac{1}{4}$	0	$\frac{5}{4}$	$c_j - z_j$

most –ve

'3/4' is the key element.

§ *Solve the given L.P.Problem by BIG-M method*

(1) Minimize $Z = -3x_1 + x_2 + x_3$ subject to,

$$x_1 - 2x_2 + x_3 \le 11$$
$$-4x_1 + x_2 + 2x_3 \ge 3$$
$$2x_1 - x_3 = -1 \quad \text{and} \quad x_1, x_2, x_3 \ge 0.$$

$\Rightarrow$ Minimize $Z = -3x_1 + x_2 + x_3 + 0x_4 + 0x_5 + Mx_6 + Mx_7$

subject to, Where M is very large +ve number.

$$x_1 - 2x_2 + x_3 + x_4 = 11$$
$$-4x_1 + x_2 + 2x_3 - x_5 + x_6 = 3$$
$$-2x_1 + x_3 + x_7 = 1$$

where $x_i \ge 0$; $i = 1, 2 \ldots.7$; x_6, x_7 are artificial variables.

Now, $n = 7$, $m = 3$ $\Rightarrow$ $n - m = 7 - 3 = 4$

x_1, x_2, x_3 and x_5 are non-basic variables but $x_4 = 11$, $x_6 = 3$ and $x_7 = 1$ is the initial basic feasible solution. We apply Big-M method.

Table-1

C_B	Vari.in basis	c_j	-3	1	1	0	0	M	M	R.Rati
		X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7	X_B / x
0	x_4	11	1	-2	1	1	0	0	0	$11/1 = 11$
M	x_5	3	-4	1	2	0	-1	1	0	$3/2$
M	x_7	1	-2	0	1	0	0	0	1	$1/1 = 1$
			$-3 + 6M$	$1 - M$	$1 - 3M$	0	M	0	0	$c_j - z$

most –ve The key element is '1'.

Table-2

C_B	Vari.in basis	c_j	-3	1	1	0	0	M	M	R.Rati
		X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7	X_B / x_3
0	x_4	10	3	-2	0	1	0	0	-1	-5
M	x_6	1	0	1	0	0	-1	1	-2	1
1	x_3	1	-2	0	1	0	0	0	1	∞
			-1	$1 - M$	0	0	M	0	$3M - 1$	$c_j - z_j$

Table-3

C_B	Vari.in basis	c_j	-3	1	1	0	0	M	M	R.Ratio
		X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7	X_B / x
0	x_4	12	3	0	0	1	-2	2	-5	4
1	x_2	1	0	1	0	0	-1	1	-2	∞
1	x_3	1	-2	0	1	0	0	0	1	$-\frac{1}{2}$
			-1	**0**	**0**	**0**	**1**	**M - 1**	**M + 1**	$c_j - z_j$

most –ve

The key element is '**3**'.

Table-4

C_B	Vari.in basis	c_j	-3	1	1	0	0	M	M	R.Ratio
		X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7	X_B / x_3
-3	x_1	4	1	0	0	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$	$-\frac{5}{3}$	
1	x_2	1	0	1	0	0	-1	1	-2	
1	x_3	9	-2	0	1	$\frac{2}{3}$	$-\frac{4}{3}$	$\frac{4}{3}$	$-\frac{7}{3}$	
			0	**0**	**0**	$\frac{1}{3}$	$\frac{1}{3}$	**M + $\frac{1}{3}$**	**M - $\frac{2}{3}$**	$c_j - z_j$

Here, all $c_j - z_j \geq 0$

$\therefore$ This is the optimum solution with $x_1 = 4$, $x_2 = 1$ and $x_3 = 9$.

$\therefore$ $\min(z) = -3(4) + 1 + 9$

$= -2.$

(2) **Maximize $Z = 3x_1 + 2x_2 + 3x_3$ subject to,**

$$2x_1 + x_2 + x_3 \leq 2$$

$$3x_1 + 4x_2 + 2x_3 \geq 8$$

and $x_1, x_2, x_3 \geq 0.$

Minimize $(-z) = -3x_1 - 2x_2 - 3x_3 + 0x_4 + 0x_5 + Mx_6$

subject to,

Where M is very large +ve number.

$$2x_1 + x_2 + x_3 + x_4 = 2$$
$$3x_1 + 4x_2 + 2x_3 - x_5 + x_6 = 8$$

and $x_i \geq 0$; $i = 1, 2 \ldots 6$; x_6 is artificial variables.

Now, $n = 6$, $m = 2 \Rightarrow n - m = 6 - 2 = 4$

x_1, x_2, x_3 and x_5 are non-basic variables but $x_4 = 2$, $x_6 = 8$ is the initial basic feasible solution.

We solve this L.P.Problem by Big-M method.

Table-1

C_B	Vari.in basis	c_j / X_B	-3 / x_1	-2 / x_2	-3 / x_3	0 / x_4	0 / x_5	M / x_6	R.Ratio X_B / x_2
0	x_4	2	2	1	1	1	0	0	$2/1 = 2$
M	x_6	8	3	4	2	0	-1	1	$8/4 = 2$
			$-3-3M$	$-2-4M$	$-3-2M$	0	M	0	$c_j - z_j$

most –ve

The key element is '1'.

Table-2

C_B	Vari.in basis	c_j / X_B	-3 / x_1	-2 / x_2	-3 / x_3	0 / x_4	0 / x_5	M / x_6	R.Ratio X_B / x_2
-2	x_2	2	2	1	1	1	0	0	
M	x_6	0	-5	0	-2	-4	-1	1	
			$1+5M$	0	$2M-1$	$2+4M$	M	0	$c_j - z_j$

Here, all $c_j - z_j \geq 0$

$\therefore$ This is the optimum solution with $x_1 = 0$, $x_2 = +2$ and $x_6 = 0$

$\therefore \max(z) = 3(0) + 2(+2) + 3(0)$

$= +4.$

Unit-03 Sem-V MAT-304

Duality in L.P.problem :-

Suppose the primal L.P.Problem given in the form

Maximize $z_x = c_1 x_1 + c_2 x_2 + \dots\dots\dots + c_n x_n$

subject to the constraints,

$$a_{11} x_1 + a_{12} x_2 + \dots\dots \dots + a_{1n} x_n \le b_1$$
$$a_{21} x_1 + a_{22} x_2 + \dots\dots \dots + a_{2n} x_n \le b_2$$
$$\dots\dots$$
$$a_{m1} x_1 + a_{m2} x_2 + \dots\dots\dots + a_{mn} x_n \le b_m$$

and $x_1, x_2, \dots\dots, x_n \ge 0$

then the corresponding dual L.P.Problem is defined as

Minimize $z_y = b_1 y_1 + b_2 y_2 + \dots\dots\dots + b_m y_m$ ← $b_m y_m$

subject to the constraints,

$$a_{11} y_1 + a_{21} y_2 + \dots\dots \dots + a_{m1} y_m \ge c_1$$
$$a_{12} y_1 + a_{22} y_2 + \dots\dots \dots + a_{m2} y_m \ge c_2$$
$$\dots\dots$$
$$a_{1n} y_1 + a_{2n} y_2 + \dots\dots\dots + a_{mn} y_m \ge c_n$$

and $y_1, y_2, \dots\dots, y_m \ge 0$

Remark :-

⇒ The Primal – Dual relationship is under.

If in primal	then in dual
(1) Objective f^n is to maximize	(1) Objective f^n is to minimize.
(2) number of variables x_j	(2) number of constraints ' j '.
(3) number of constraints ' i '	(3) number of variables y_i.
(4) '≤' type constraints	(4) '≥' type constraints.
(5) variable x_j unrestricted in sign	(5) constraint ' j ' is ' = ' type.
(6) constraint ' i ' is ' = ' type	(6) variable x_i unrestricted in sign.

⇒ In the both Primal and Dual Problem the variables are non-negative and the constraints are inequalities such problem are called the symmetric dual linear programming problem.

⇒ For maximization problem all constraints be with '≤' type and for minimization problem all constraints be with '≥' type.

⇒ **The Dual of the dual**

$$\text{Min } z = x_1 - 3x_2 - 2x_3$$

Subject to,

$$3x_1 - x_2 + 2x_3 \le 7$$
$$3x_1 - 4x_2 \ge 12$$
$$-4x_1 + 3x_2 + 8x_3 = 10$$

and $x_1, x_2, x_3 \ge 0$.

i.e. $\text{Min } z = x_1 - 3x_2 - 2x_3$

Subject to,

$$-3x_1 + x_2 - 2x_3 \ge -7$$
$$3x_1 - 4x_2 \ge 12$$
$$-4x_1 + 3x_2 + 8x_3 = 10$$

and $x_1, x_2, x_3 \ge 0$.

Thus, the dual of the dual is a primal.

Example :- *Find the Dual of the Primal. Solve both primal and the dual by the Simplex or Big-M method..* 2013

$$\text{Max } z = 3x_1 + 2x_2$$

Subject to,

$$2x_1 + x_2 \le 5$$
$$x_1 + x_2 \le 3$$

and $x_1, x_2 \ge 0$.

⇒ **Dual**

$$\text{Min } w = 5y_1 + 3y_2$$

Subject to,

$$2y_1 + y_2 \ge 3$$
$$y_1 + y_2 \ge 2$$

and $y_1, y_2 \ge 0$.

⇒ **Simplex method for primal.**

$$\text{Min } (-z) = -3x_1 - 2x_2$$

Subject to,

$$2x_1 + x_2 + x_3 = 5$$
$$x_1 + x_2 + x_4 = 3$$

and $x_1, x_2, x_3, x_4 \ge 0$.

Now, $n = 4$, $m = 2 \Rightarrow n - m = 4 - 2 = 2$

Here, x_1 and x_2 are non-basic variables and $x_3 = 5$ and $x_4 = 3$ are basic variables. Which is the initial basic feasible solution.

The initial simplex table is presented as follows.

Table : - 1

C_B	Vari.in basis	C_j / X_B	-3 / x_1	-2 / x_2	0 / x_3	0 / x_4	R.Ratio X_B / x_1
0	x_3	5	2	1	1	0	$5/2$
0	x_4	3	1	1	0	1	$3/1 = 3$
$-z = \sum C_{B_i} X_{B_i} = 0$			-3	-2	0	0	$c_j - z_j$

most –ve

Table : - 2

C_B	Vari.in basis	C_j / X_B	-3 / x_1	-2 / x_2	0 / x_3	0 / x_4	R.Ratio X_B / x_2
-3	x_1	$\frac{5}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	5
0	x_4	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	1	1
$-z = \sum C_{B_i} X_{B_i} = -\frac{15}{2}$			0	$-\frac{1}{2}$	$\frac{3}{2}$	0	$c_j - z_j$

most –ve

Table : - 3

C_B	Vari.in basis	C_j / X_B	-3 / x_1	-2 / x_2	0 / x_3	0 / x_4	R.Ratio
-3	x_1	2	1	0	1	-1	
-2	x_2	1	0	1	-1	2	
$-z = \sum C_{B_i} X_{B_i} = -8$			0	0	1	1	$c_j - z_j$

Here, all $c_j - z_j \geq 0$

$\therefore$ This is the optimum solution with $x_1 = 2,\ x_2 = 1.$

$$\min(-z) = -3(2) - 2(1)$$
$$= -8$$

$\therefore \max(z) = -\min(-z) = 8.$

⇒ **Big-M method for Dual.**

$$\text{Min } w = 5y_1 + 3y_2$$

Subject to,

$$2y_1 + y_2 \geq 3$$
$$y_1 + y_2 \geq 2$$

and $y_1, y_2 \geq 0$.

⇒ $\text{Min } w = 5y_1 + 3y_2 + 0y_3 + 0y_4 + My_5 + My_6$

Where M is very large +ve number.

subject to,

$$2y_1 + y_2 - y_3 + y_5 = 3$$
$$y_1 + y_2 - y_4 + y_6 = 2$$

where $y_i \geq 0$; $i = 1, 2 \ldots 6$ & y_5, y_6 are artificial variables.

Now, $n = 6$, $m = 2$ ⇒ $n - m = 6 - 2 = 4$

y_1, y_2, y_3 and y_4 are non-basic variables but $y_5 = 3$, $y_6 = 2$ is the initial basic feasible solution. We apply Big-M method.

Table- 1

C_B	Vari.in basis	c_j / X_B	5 / y_1	3 / y_2	0 / y_3	0 / y_4	M / y_5	M / y_6	R.Ratio X_B / y_1
M	y_5	3	2	1	−1	0	1	0	3 / 2
M	y_6	2	1	1	0	−1	0	1	2 / 1 = 2
			5 − 3M	3 − 2M	M	M	0	0	$c_j - w_j$

most −ve ‘2’ is the key element.

Table- 2

C_B	Vari.in basis	c_j / X_B	5 / y_1	3 / y_2	0 / y_3	0 / y_4	M / y_5	M / y_6	R.Ratio X_B / y_1
5	y_1	$\frac{3}{2}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	3
M	y_6	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	−1	$-\frac{1}{2}$	1	1
			0	$\frac{1}{2} - \frac{M}{2}$	$\frac{5}{2} - \frac{M}{2}$	M	$-\frac{5}{2} + \frac{3M}{2}$	0	$c_j - w_j$

most −ve $\frac{1}{2}$ is the key element.

Theorem :- *Prove that the dual of the dual is the primal.* 2013

$\Rightarrow$ Let the primal problem be

$$\text{Minimize } z = CX$$

subject to $AX \geq b$; $X \geq 0$. .... (1)

where A is 'm x n' matrix, b is 'm x 1' matrix, X is '~~1 x n~~ $n \times 1$' matrix & C is '1 x n' matrix.

The dual of the primal problem be

$$\text{Maximize } w = b^T Y$$

$$\text{subject to } A^T Y \leq C^T \text{ ; } Y \geq 0.$$

But, Maximize(z) = – Minimize(– z)

$\therefore$ This dual can be written as,

$$-\text{Minimize}(-w) = b^T Y$$

$$\text{subject to } A^T Y \leq C^T \text{ ; } Y \geq 0.$$

$$\therefore \text{ Minimize}(-w) = -b^T Y$$

$$\text{subject to } -A^T Y \geq -C^T \text{ ; } Y \geq 0.$$

Now, The dual of this I.P.Problem is

$$\text{Maximize}(-z) = (-C^T)^T X = -CX$$

$$\text{subject to } (-A^T)^T X \leq (-b^T)^T \text{ ; } X \geq 0.$$

i.e. $-\text{Maximize}(-z) = CX$

$$\text{subject to } -AX \leq -b \text{ ; } X \geq 0.$$

i.e. $\text{Minimize}(z) = CX$

$$\text{subject to } AX \geq b \text{ ; } X \geq 0.$$

Which is the primal.

Thus, the dual of the dual is the primal.

§ *Explain the Dual Simplex method for solving L.P.Problem.*

Step-1

Convert all the inequalities of '$\geq$' type to '$\leq$' type.

Step-2

Obtain an initial basic solution to the L.P.Problem and put the solution in the initial dual simplex table.

Step-3

Calculate the value of $c_j - z_j$ and examine,

(i) If all $c_j - z_j$ and basic variables x_{Bi} are non-negative ; for all i and j then the optimal basic feasible solution has been obtain.

(ii) If all $c_j - z_j$ are non-negative and at least one basic variable say x_{Bi} is negative the go to step-4.

(iii) If at least one $c_j - z_j$ is negative then method is not applicable to the given L.P.Problem.

Step-4

Select the most negative of x_{Bi}'s the corresponding basic variable then leave the basis. Suppose the basic variable x_K is the most negative so x_K will leave the basis.

Step-5

Test the nature of x_{k_j} ; j = 1, 2, ...n (n = number of columns)

(i) If all x_{k_j} are non-negative then there does not exist any feasible solution to the given L.P.Problem.

(ii) If atleast one x_{k_j} is negative then compute the replacement ratio

$\dfrac{c_j - z_j}{x_{k_j}}$ where $x_{k_j} < 0$ And choose the maximum of these values.

Suppose the corresponding variable say x_{k_r} then compute the

$$\max \left\{ \frac{c_j - z_j}{x_{k_j}} \right\} ; j = 1, 2, \ldots n.$$

The variable x_k will leave the basis and x_r will enter into the basis.

The element that lies at the intersection of the raw x_k and the column x_r is the key element. Convert it into unity and all the other elements of the column x_r into zero.

Step-6 Test the new dual Simplex table for optimality. Repeat the process until either an optimum feasible solution has been obtain or there is an indication of the non-existence of a feasible solution.

Remark :- (i) The Dual Simplex method applies to the problem which start the optimum but infeasible solution. of the dual is the primal.
(ii) In the Dual Simplex method all constraints to be convert into $\leq$ type either maximize or minimize problem.

the problem is
Example – Write down the Dual of the following L.P.Problem solve it by Simplex method. Find all the optimum value of the primal.

$$\text{Min } z = 2x_1 + 3x_2$$

Subject to,

$$3x_1 + 2x_2 \geq 6$$
$$2x_1 + 5x_2 \geq 10$$
$$\text{and } x_1, x_2 \geq 0.$$

$\Rightarrow$ Dual of the given primal is

$$\text{Maximize } w = 6y_1 + 10y_2$$

Subject to,

$$3y_1 + 2y_2 \leq 2$$
$$2y_1 + 5y_2 \leq 3$$
$$\text{and } y_1, y_2 \geq 0.$$

$\therefore$ Minimize $(-w) = -6y_1 - 10y_2$

Subject to,

$$3y_1 + 2y_2 + y_3 = 2$$
$$2y_1 + 5y_2 + y_4 = 3$$

and $y_i \geq 0$; $i = 1, 2, 3, 4$ & y_4 ; y_5 – slack variables.

Now, $n = 5$, $m = 2$ $\Rightarrow$ $n - m = 5 - 2 = 3$

y_1, y_2 are non-basic variables and basic variables are $y_3 = 2$, $y_4 = 3$ which is the initial basic feasible solution.

Table- 1

C_B	Vari.in basis	c_j	-6	-10	0	0	R.Ratio
		X_B	y_1	y_2	y_3	y_4	X_B / y_2
0	y_3	2	3	2	1	0	$2/2 = 1$
0	y_4	3	2	$\boxed{5}$	0	1	$3/5$
			-6	-10	0	0	$c_j - w_j$

most –ve

'5' is the key element.

Table-2

C_B	Vari.in basis	c_j	-6	-10	0	0	R.Ratio
		X_B	y_1	y_2	y_3	y_4	X_B / y_1
0	y_3	$\frac{4}{5}$	$\frac{11}{5}$	0	1	$-\frac{2}{5}$	$4/11$
-10	y_2	$\frac{3}{5}$	$\frac{2}{5}$	1	0	$\frac{1}{5}$	$3/2$
			-2	0	0	2	$c_j - w_j$

most –ve

$\therefore \frac{11}{5}$ is the key element.

Table-3

C_B	Vari.in basis	c_j	-6	-10	0	0	R.Ratio
		X_B	y_1	y_2	y_3	y_4	X_B / y_1
-6	y_1	$\frac{4}{11}$	1	0	$\frac{5}{11}$	$-\frac{2}{11}$	
-10	y_2	$\frac{5}{11}$	0	1	$-\frac{2}{11}$	$\frac{3}{11}$	
			0	0	$\frac{10}{11}$	$\frac{18}{11}$	$c_j - w_j$

Here, all $c_j - w_j \geq 0$

$\therefore$ This is the optimum solution with $y_1 = \frac{4}{11}, y_2 = \frac{5}{11}$.

$$\max w = 6\left(\frac{4}{11}\right) + 10\left(\frac{5}{11}\right)$$

$$= \frac{74}{11}$$

For the primal,

We have $x_1 = \frac{10}{11}, x_2 = \frac{18}{11}$

$$\min z = 2\left(\frac{10}{11}\right) + 3\left(\frac{18}{11}\right)$$

$$= \frac{74}{11}$$

L.P.Problem.

Example – Solve the given L.P.Problem.

$$\text{Max } z = -3x_1 - 2x_2$$

Subject to,

$$x_1 + x_2 \geq 1$$
$$x_1 + x_2 \leq 7$$
$$x_1 + 2x_2 \geq 10$$
$$x_2 \leq 3 \text{ and } x_1, x_2 \geq 0.$$

⇒ The given L.P.Problem is

$$\text{Max } z = -3x_1 - 2x_2$$

Subject to,

$$-x_1 - x_2 \leq -1$$
$$x_1 + x_2 \leq 7$$
$$-x_1 - 2x_2 \leq -10$$
$$x_2 \leq 3 \text{ and } x_1, x_2 \geq 0.$$

$$\therefore \text{ Min }(-z) = 3x_1 + 2x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6$$

Subject to,

$$-x_1 - x_2 + x_3 = -1$$
$$x_1 + x_2 + x_4 = 7$$
$$-x_1 - 2x_2 + x_5 = -10$$
$$x_2 + x_6 = 3 \text{ and } x_1, x_2 \geq 0.$$

Here, x_1 and x_2 are non-basic variables and $x_3 = -1$, $x_4 = 7$ $x_5 = -10$ and $x_6 = 3$ are basic variables. Which is the initial basic solution but not feasible. So we apply Dual Simplex method.

Table-1

C_B	Vari.in basis	c_j / X_B	3 / x_1	2 / x_2	0 / x_3	0 / x_4	0 / x_5	0 / x_6	
0	x_3	– 1	– 1	– 1	1	0	0	0	
0	x_4	7	1	1	0	1	0	0	
0	x_5	– 10	– 1	$\boxed{-2}$	0	0	1	0	
0	x_6	3	0	1	0	0	0	1	
			3	2	0	0	0	0	$c_j - z_j$

Here,

All $c_j - z_j \geq 0$ and the value of basic variables $x_3 = -1$, $x_4 = 7$

$x_5 = -10$ and $x_6 = 3$

$\therefore$ This is an optimum solution but not feasible.

The most negative basic variable is $x_5 = -10$.

Consider the ratio $\frac{c_j - z_j}{x_{5_j}}$; where $x_{5_j} < 0$; $j = 1, 2$.

$$\max\left\{\frac{c_j - z_j}{x_{5_j}}\right\} = \max\left\{\frac{3}{-1}, \frac{2}{-2}\right\} = \max\{-3, -1\} = -1$$

i.e. the maximum ratio is -1 and it occur in the column of x_2.

So x_5 will leave the basis and x_2 will enter into the basis.

Table-2

C_B	Vari.in basis	c_j / X_B	3 / x_1	2 / x_2	0 / x_3	0 / x_4	0 / x_5	0 / x_6	
0	x_3	4	$-\frac{1}{2}$	0	1	0	$-\frac{1}{2}$	0	
0	x_4	2	$\frac{1}{2}$	0	0	1	$\frac{1}{2}$	0	
2	x_2	5	$\frac{1}{2}$	1	0	0	$-\frac{1}{2}$	0	
0	x_6	-2	$\boxed{-\frac{1}{2}}$	0	0	0	$\frac{1}{2}$	1	
			2	0	0	0	1	0	$c_j - z_j$

Here,

All $c_j - z_j \geq 0$ and the value of basic variables $x_2 = 5$, $x_3 = 4$,

$x_4 = 2$ and $x_6 = -2$.

$\therefore$ This is an optimum solution but not feasible.

The most negative basic variable is $x_6 = -2$

Consider the ratio $\frac{c_j - z_j}{x_{6_j}}$; where $x_{6_j} < 0$; $j = 1$.

$$\max\left\{\frac{c_j - z_j}{x_{6_j}}\right\} = \max\left\{\frac{2}{-1/2}\right\} = \max\{-4\} = -4$$

i.e. the maximum ratio is -4 and it occur in the column of x_1.

So x_6 will leave the basis and x_1 will enter into the basis.

Table-3

C_B	Vari.in basis	c_j / X_B	3 / x_1	2 / x_2	0 / x_3	0 / x_4	0 / x_5	0 / x_6	
0	x_3	6	0	0	1	0	-1	-1	
0	x_4	0	0	0	0	1	1	1	
2	x_2	3	0	1	0	0	0	1	
3	x_1	4	1	0	0	0	-1	-2	
			0	0	0	0	3	4	$c_j - z_j$

Here,
All $c_j - z_j \geq 0$ and all basic variables are positive.
$\therefore$ This is an optimum basic feasible solution with $x_1 = 4$, $x_2 = 3$, $x_3 = 6$ and $x_4 = 0$.

$\therefore$ Max z $= -3(4) - 2(3)$
$= -18.$

Example – Use the Dual Simplex method to solve the given L.P.Problem.

Min $z = x_1 + x_2$

Subject to,

$$2x_1 + x_2 \geq 4$$
$$x_1 + 7x_2 \geq 7 \quad \text{and } x_1, x_2 \geq 0.$$

$\Rightarrow$ The given L.P.Problem is

Min $z = x_1 + x_2$

Subject to,

$$-2x_1 - x_2 \leq -4$$
$$-x_1 - 7x_2 \leq -7 \quad \text{and } x_1, x_2 \geq 0.$$

$\therefore$ Min $z = x_1 + x_2 + 0S_1 + 0S_4$

Subject to,

$$-2x_1 - x_2 + S_1 = -4$$
$$-x_1 - 7x_2 + S_2 = -7 \quad \text{and } x_1, x_2 \geq 0.$$

Here, x_1 and x_2 are non-basic variables and $x_3 = -4$, $x_4 = -7$ are basic variables. Which is the initial basic solution but not feasible.

$\therefore$ we apply Dual Simplex method.

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A2

Table-1

C_B	Vari.in basis B	$c_j \rightarrow$ / X_B basic	1 / x_1	1 / x_2	0 / s_1	0 / s_2	
0	s_1	-4	-2	-1	1	0	
0	s_2	-7	-1	$\boxed{-7}$	0	1	
		$z_j \rightarrow$	0	0	0	0	
		$c_j - z_j \rightarrow$	1	1	0	0	$c_j - z_j$
		$\frac{c_j - z_j}{s_2}$	-1	$-\frac{1}{7}$ max	0	0	

Here,

All $c_j - z_j \geq 0$ and the value of basic variables $x_3 = -4$, $x_4 = -7$.

$\therefore$ This is an optimum solution but not feasible.

The most negative basic variable is $x_4 = -7$. $\therefore$ max, $\frac{c_j - z_j}{s_2} = -\frac{1}{7}$

Consider the ratio $\frac{c_j - z_j}{s_2}$; where $x_{4j} < 0$; $j = 1, 2$.

$$\max\left\{\frac{c_j - z_j}{x_{4j}}\right\} = \max\left\{\frac{1}{-1}, \frac{1}{-7}\right\} = \max\left\{-1, -\frac{1}{7}\right\} = -\frac{1}{7}$$

i.e. the maximum ratio is $-\frac{1}{7}$ and it occur in the column of x_2.

So x_4 will leave the basis and x_2 will enter into the basis.

Table-2

R2'(-1/7)

R2(-1/7)

remove S_2 enter x_2

C_B	Vari.in basis B	$c_j \rightarrow$ / X_B basic	1 / x_1	1 / x_2	0 / s_1	0 / s_2	
0	s_1	-3	$\boxed{-\frac{13}{7}}$	0	1	$-\frac{1}{7}$	
1	x_2	1	$\frac{1}{7}$	1	0	$-\frac{1}{7}$	
		$z_j \rightarrow$	$\frac{1}{7}$	1	0	$-\frac{1}{7}$	
		$c_j - z_j \rightarrow$	$\frac{6}{7}$	0	0	$\frac{1}{7}$	$c_j - z_j$
		$\frac{c_j - z_j}{s_1}$	$-\frac{6}{13}$ max	0	0	-1	

Here,

All $c_j - z_j \geq 0$ and the value of basic variables $x_2 = 1$, $x_3 = -3$

$\therefore$ This is an optimum solution but not feasible.

The most negative basic variable is $x_3 = -3$.

Consider the ratio $\frac{c_j - z_j}{s_1}$; where $x_{3j} < 0$; $j = 1, 4$.

(A3)

$$\max\left\{\frac{c_j - z_j}{x_{3j}}\right\} = \max\left\{\frac{6/7}{-13/7}, \frac{1/7}{-1/7}\right\} = \max\left\{-\frac{6}{13}, -1\right\} = -\frac{6}{13}$$

i.e. the maximum ratio is $-\frac{6}{13}$ and it occur in the column of x_1.

So x_3 will leave the basis and x_1 will enter into the basis.

Table-3

C_B	Vari.in basis B	$c_j \rightarrow$ basic X_B	1 x_1	1 x_2	0 s_1	0 s_2	
1	x_1	$\frac{21}{13}$	1	0	$-\frac{7}{13}$	$\frac{1}{13}$	
1	x_2	$\frac{10}{13}$	0	1	$\frac{1}{13}$	$-\frac{2}{13}$	
		$z_j \rightarrow$	1	1	$-\frac{6}{13}$	$-\frac{1}{13}$	$c_j - z_j$
		$c_j - z_j \rightarrow$	0	0	$\frac{6}{13}$	$\frac{1}{13}$	

$R_1\left(-\frac{7}{13}\right)$

remove s_1

enter x_1

$R_{12}\left(\frac{1}{13}\right)$

Here,

All $c_j - z_j \geq 0$ and all basic variables are positive.

$\therefore$ This is an optimum basic feasible solution with $x_1 = \frac{21}{13}$ and

$x_2 = \frac{10}{13}$.

$\therefore \text{Min } z = x_1 + x_2$

$= \frac{21}{13} + \frac{10}{13}$

$= \frac{31}{}$

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Integer Programming Problem

In many practical problems the decision variables make sense only if they have integer value.

For example, if the variables are number of buses on different routes in the town or the number of bank branches in different regions of country fraction answer have no meaning.

Mathematical programming subject to the constraints that the variables are integer is called integer programming problem.

Minimize $z = F(x)$

Subject to

$Ax = B$; $x \geq 0$ and x is an integer.

Example – Solve the given L.P.Problem.

Max $z = x_1 + x_2$

Subject to,

$3x_1 + 2x_2 \leq 5$

$x_2 \leq 2$ **and** $x_1, x_2 \geq 0$; x_1 **and** x_2 **are integers.**

By cutting plane method.

$\Rightarrow$ Here, $\max(z) = -\min(-z)$

$\therefore \min(-z) = -x_1 - x_2 + 0x_3 + 0x_4$

subject to,

$3x_1 + 2x_2 + x_3 = 5$

$x_2 + x_4 = 2$

and $x_i \geq 0$; $i = 1, 2, 3, 4$

Now, $n = 4$, $m = 2$ $\Rightarrow$ $n - m = 4 - 2 = 2$

x_1, x_2 are non-basic variables and $x_3 = 5$, $x_4 = 2$ are basic variables. This is the initial basic feasible solution.

Table-1

C_B	Vari.in basis	C_j / X_B	-1 / x_1	-1 / x_2	0 / x_3	0 / x_4	R.Ratio X_B / x_i
0	x_3	5	3	2	1	0	5 / 3
0	x_4	2	0	1	0	1	∞
			-1	-1	0	0	$c_j - z_j$

most –ve

'3' is the key element.

Table-2

C_B	Vari.in basis	C_j / X_B	-1 / x_1	-1 / x_2	0 / x_3	0 / x_4	R.Ratio X_B/x_2
-1	x_1	$\frac{5}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	0	5/2
0	x_4	2	0	1	0	1	2 →
			0	$-\frac{1}{3}$	$\frac{1}{3}$	0	$c_j - z_j$

most –ve

'1' is the key element.

Table-3

C_B	Vari.in basis	C_j / X_B	-1 / x_1	-1 / x_2	0 / x_3	0 / x_4	R.Ratio
-1	x_1	$\frac{1}{3}$	1	0	$\frac{1}{3}$	$-\frac{2}{3}$	
-1	x_2	2	0	1	0	1	
			0	0	$\frac{1}{3}$	$\frac{1}{3}$	$c_j - z_j$

Here,

All $c_j - z_j \geq 0$ So this is an optimum basic feasible solution with $x_1 = \frac{1}{3}$, $x_2 = 2$.

But x_1 is not an integer.

ceiling function
→ નાનામાં નાના [illegible]

∴ The constraint corresponding to $x_1 = \frac{1}{3}$ is

$$x_1 + 0x_2 + \frac{1}{3}x_3 - \frac{2}{3}x_4 = \frac{1}{3}$$

$$\therefore x_1 + \frac{1}{3}x_3 - \frac{2}{3}x_4 = \frac{1}{3}$$

$$\therefore x_1 + \left(\left[\frac{1}{3}\right] + \frac{1}{3}\right)x_3 + \left(\left[-\frac{2}{3}\right] + \frac{1}{3}\right)x_4 = \left[\frac{1}{3}\right] + \frac{1}{3}$$

$$\therefore x_1 + \left(0 + \frac{1}{3}\right)x_3 + \left(-1 + \frac{1}{3}\right)x_4 = 0 + \frac{1}{3}$$

$$\therefore \underbrace{x_1 - x_4}_{S_2} = \frac{1}{3} - \left(\frac{1}{3}x_3 + \frac{1}{3}x_4\right)$$

i.e. $x_1 - x_4 \leq 0$ is an integer.

$$\therefore \frac{1}{3} - \left(\frac{1}{3}x_3 + \frac{1}{3}x_4 \right) \le 0$$

$$\therefore \frac{1}{3} \le \frac{1}{3}x_3 + \frac{1}{3}x_4$$

$$\therefore -\frac{1}{3} \ge -\frac{1}{3}x_3 - \frac{1}{3}x_4$$

$$\text{i.e. } -\frac{1}{3}x_3 - \frac{1}{3}x_4 \le -\frac{1}{3}$$

$$\therefore -\frac{1}{3}x_3 - \frac{1}{3}x_4 + x_5 = -\frac{1}{3}$$

Where x_5 is called Gomorian slack variable.

Table-4

C_B	Vari. in basis	C_j / X_B	-1 / x_1	-1 / x_2	0 / x_3	0 / x_4	0 / x_5	R.Ratio
-1	x_1	$\frac{1}{3}$	1	0	$\frac{1}{3}$	$-\frac{2}{3}$	0	
-1	x_2	2	0	1	0	1	0	
0	x_5	$-\frac{1}{3}$	0	0	$\boxed{-\frac{1}{3}}$	$-\frac{1}{3}$	1	
			0	0	$\frac{1}{3}$	$\frac{1}{3}$	0	$c_j - z_j$

Here,

All $c_j - z_j \ge 0$ and the value of basic variables $x_1 = \frac{1}{3}$, $x_2 = 2$ and $x_5 = -\frac{1}{3}$.

$\therefore$ This is an optimum solution but not feasible.
So we apply dual Simplex method.

The most negative basic variable is $x_5 = -\frac{1}{3}$.

Consider the ratio $\frac{c_j - z_j}{x_{5j}}$; where $x_{5j} < 0$; $j = 3, 4$.

$$\max \left\{ \frac{c_j - z_j}{x_{5j}} \right\} = \max \left\{ \frac{1/3}{-1/3}, \frac{1/3}{-1/3} \right\} = \max \{-1, -1\} = -1$$

i.e. maximum ratio is -1 and it occur in the column of x_3 and x_4.
We select column x_3. So x_5 will leave the basis and x_3 will

enter into the basis.

Table-5

C_B	Vari.in basis	C_j	-1	-1	0	0	0	
		X_B	x_1	x_2	x_3	x_4	x_5	
-1	x_1	0	1	0	0	-1	$\cancel{0}$ 1	
-1	x_2	2	0	1	0	1	0	
0	x_3	1	0	0	1	1	-3	
			0	0	0	0	1	$c_j - z_j$

Here,

All $c_j - z_j \geq 0$ and the value of basic variables $x_1 = 0, x_2 = 2$ and $x_3 = 1$.

$\therefore$ This is an optimum basic feasible solution.

$$\therefore \text{Max } z = x_1 + x_2$$
$$= 0 + 2$$
$$= 2$$

TRANSPORTATION PROBLEM

The transportation problem is a special case of the linear programming problem. It deals with the situation in which a single product is transported from *Sources* to *Destinations*. The objective is to determine the amount of a single product to be transported from each source to each destination so that the total transportation cost is minimum.

⇒ ***Transportation Problem : -***

The transportation problem applies to situations in which a single product is to be transported from several sources (origin or supply or capacity centres) to several sinks (demand or destination or requirement centres).

Let there be m sources $S_1, S_2, \ldots, S_m$ with a_i (i = 1, 2, ….m) available suppliers or capacity at each source 'i' to be allocated among n-destinations $D_1, D_2, \ldots, D_n$ with b_j (j = 1, 2, …..,n) specified requirements at each destination 'j' for each route.

Let C_{ij} be the cost of transportation per unit from source 'i' to destination 'j'. If x_{ij} represent the units shipped per route from source 'i' to destination 'j', then the problem is to determine the transportation schedule so as to minimize the total transportation cost satisfying supply and demand conditions.

$$\text{Minimize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} x_{ij}$$

Subject to,

$$\sum_{j=1}^{n} x_{ij} = a_i \; ; i = 1, 2, \ldots m. \text{ (capacity constraints)}$$

$$\sum_{i=1}^{m} x_{ij} = b_j \; ; j = 1, 2, \ldots n. \text{ (requirement constraints)}$$

and $x_{ij} \geq 0$, for all i, j.

The transportation problem presented in the form of a rectangular array as shown under.

To → From ↓	D_1	D_2	.	D_n	Supply a_i
S_1	C_{11} x_{11}	C_{12} x_{12}		C_{1n} x_{1n}	a_1
S_2	C_{21} x_{21}	C_{22} x_{22}	.	C_{2n} x_{2n}	a_2
⋮	⋮	⋮		⋮	
S_m	C_{m1} x_{m1}	C_{m2} x_{m2}		C_{mn} x_{mn}	a_m
Demand b_j	b_1	b_2		b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

⇒ *Rim requirements :* - The various origin capacities and various destination requirements are listed in the right most column (outer column) and in the bottom row respectively. These are called the rim requirements.

⇒ ***Balanced and unbalanced transportation problem :*** - If the total capacity equals the total requirement then the problem is called ***balanced transportation problem*** otherwise ***unbalanced transportation problem.***

⇒ ***A necessary and sufficient condition for the existence of a feasible solution to the transportation problem is*** $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.

⇒ ***Feasible solution :*** - A set of non-negative allocations i.e. $x_{ij} \geq 0$ is called feasible solution.

⇒ ***Basic feasible solution :*** - A feasible solution to m-origins and n-destinations problem is said to be basic feasible solution if the number of positive allocations is $m + n - 1$. i.e. one less than the sum number of rows and columns.

⇒ ***Degenerate basic feasible solution :*** - If the number of positive allocations in a basic feasible solution is less than $m + n - 1$ then the solution is said to be degenerate basic feasible solution. Otherwise is said to be non-degenerate basic feasible solution.

⇒ ***Optimum solution :*** - A feasible solution is called an optimum solution if it is minimizes the total transportation cost.

⇒ ***North-West Corner Method :* - (N.W.C.M.)**

Step – 1

Starting with the cell at the upper left (North-West) corner of the transportation matrix and allocate (to allot) as much as possible equal to the minimum of the rim values for the first row and first column. i.e min(a_1, b_1).

Step – 2

(1) If allocation made in step – 1 is equal to the capacity of the first source (first row) then move down to the first cell in the second row and first column and apply step – 1 again for next allocation.

(2) If allocation made in step – 1 is equal to the requirement of the first destination (first column) then move horizontally to the second cell in the first row and second column and apply step – 1 again for next allocation.

Step – 3

Continue the procedure [illegible] capacity [illegible]

⇒ *Least Cost Method : - (L.C.M.)*

Step – 1

Select the cell with the minimum unit transportation cost among all the rows and columns of the transportation table and allocate as much as possible to this cell and eliminate (lined out) that row or column in which either capacity or require -ment is exhausted (to make empty).

In case the minimum cost cell is not unique, then select the cell where maximum allocation can be made.

Step – 2

Select the cell with the next minimum unit transportation cost among all the rows and columns of the transportation table and allocate as much as possible to this cell and eliminate that row or column in which either capacity or requirement is exhausted.

Step – 3

Repeat the procedure until the entire available capacity at various sources and requirement at the various destinations is satisfies. The solution so obtained need not be non-degenerate.

⇒ *Vogel's Approximation Method : - (V.A.M.)*

Step – 1

Calculate penalties by taking the difference between the minimum and next minimum unit transportation cost in each row and column. Put them along the transportation table by enclosing them in the parenthesis against the respective rows and columns. This difference indicates the penalty or extra cost which has to be paid if one fails to allocate to the cell with minimum transportation cost.

Step – 2

Select the row or column with the large penalty among all the rows and columns. Let the largest penalty corresponds to the i^{th} row and c_{ij} be the smallest cost in the i^{th} row. Allocate the largest possible amount $x_{ij} = \min(a_i, b_j)$ in the cell (i, j) and cross out the i^{th} row or j^{th} column in the usual manner.

If there is a tie in the values of penalties, then it can be broken by selecting the cell where maximum allocation can be made.

Step – 3

Recalculate the row and column penalties for the reduced transportation table and then go to step – 2. Repeat the procedure until the rim requirements are satisfied.

Example : - Determine an initial basic feasible solution to the following T.P. using North-West Corner Method.

	D_1	D_2	D_3	D_4	Supply a_i
O_1	6	4	1	5	14
O_2	8	9	2	7	16
O_3	4	3	6	2	5
Demand b_j	6	10	15	4	35

⇒ The given transportation problem is balanced.

	D_1	D_2	D_3	D_4	Supply a_i
O_1	6 $\boxed{6}$	4 $\boxed{8}$	1	5	~~14~~ ~~8~~ 0
O_2	8	9 $\boxed{2}$	2 $\boxed{14}$	7	~~16~~ ~~14~~ 0
O_3	4	3	6 $\boxed{1}$	2 $\boxed{4}$	~~5~~ ~~4~~ 0
Demand b_j	~~6~~ 0	~~10~~ ~~2~~ 0	~~15~~ ~~1~~ 0	~~4~~ 0	35

The total transportation cost Z,

$= (6 \times 6) + (4 \times 8) + (9 \times 2) + (2 \times 14) + (6 \times 1) + (2 \times$

$= 36 + 32 + 18 + 28 + 6 + 8$

$= 128$

Remark :-

(1) The number of basic variables means the number of positive allocation.

$= m + n - 1 = 3 + 4 - 1 = 6$

(2) By moving in the same manner horizontally and vertically as successive requirement and supply are met we ensure that the solution is feasible.

Example : - Determine an initial basic feasible solution to the following T.P. using the (1) North-West Corner Method

(2) Least Cost Method

(3) Vogel's Approximation Method

Wear houses / Factory	W_1	W_2	W_3	W_4	Factory capacity a_i
F_1	19	30	50	10	7
F_2	70	30	40	60	9
[illegible]	[illegible]	[illegible]	70	[illegible]	[illegible]
[illegible]					

(1) North-West Corner Method :-

Wear houses / Factory	W_1	W_2	W_3	W_4	Factory capacity a_i
F_1	19 $\boxed{5}$	30 $\boxed{2}$	50	10	$\cancel{7}$ $\cancel{2}$ 0
F_2	70	30 $\boxed{6}$	40 $\boxed{3}$	60	$\cancel{9}$ $\cancel{3}$ 0
F_3	40	8	70 $\boxed{4}$	20 $\boxed{14}$	$\cancel{18}$ $\cancel{14}$ 0
Wearhouse Requirement b_j	$\cancel{5}$ 0	$\cancel{8}$ $\cancel{6}$ 0	$\cancel{7}$ $\cancel{4}$ 0	$\cancel{14}$ 0	34

The number of basic variables means the number of positive allocations $= m + n - 1 = 3 + 4 - 1 = 6$

ટાઇમમાં
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The total transportation cost Z,

$$= (19 \times 5) + (30 \times 2) + (30 \times 6) + (40 \times 3) + (70 \times 4) + (20 \times 14)$$

$$= 95 + 60 + 180 + 120 + 280 + 280$$

$$= 1015$$

(2) Least Cost Method :-

Here the lowest cost is '8' in the cell (F_3, W_2) so we allocate as much as possible equal to the minimum of the rim values. i.e. here we allocate '8' and eliminate the column W_2 as the requirement of W_2 is exhausted.

The next lowest cost is '10' in the cell (F_1, W_4) so we allocate as much as possible equal to the minimum of the rim values. i.e. here we allocate '7' and eliminate the row F_1 as the capacity of F_1 is exhausted.

Repeat the procedure until the rim requirements are satisfies.

Wear houses / Factory	W_1	W_2	W_3	W_4	Factory capacity a_i
F_1	19	30	50	10 $\boxed{7}$	$\cancel{7}$ 0
F_2	70 $\boxed{2}$	30	40 $\boxed{7}$	60	$\cancel{9}$ $\cancel{2}$ 0
F_3	40 $\boxed{3}$	8 $\boxed{8}$	70	20 $\boxed{7}$	$\cancel{18}$ $\cancel{10}$ $\cancel{3}$ 0
Wearhouse Requirement b_j	$\cancel{5}$ $\cancel{2}$ [illegible]	[illegible]	[illegible] 0	$\cancel{14}$ $\cancel{7}$ 0	

[illegible]
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લખવાનું
અનુક્રમ

The number of basic variables means the number of positive allocations = m + n – 1 = 3 + 4 – 1 = 6

The total transportation cost Z,

$$= (10 \times 7) + (70 \times 2) + (40 \times 7) + (40 \times 3) + (8 \times 8) + (20 \times 7)$$

$$= 70 + 140 + 280 + 120 + 64 + 140$$

$$= 814$$

(3) Vogel's Approximation Method :-

First we calculate penalties by taking the difference between the minimum and next minimum unit transportation cost in each row and column. Put them along the the transportation table by enclosing them in the parenthesis against the respective rows and columns.

Table – 1

Wear houses / Factory	W_1	W_2	W_3	W_4	Factory capacity a_i	
F_1	19	30	50	10	7	(9)
F_2	70	30	40	60	9	(10)
F_3	40	8 $\boxed{8}$	70	20	~~18~~ 10	(12)
Wearhouse Requirement b_j	5	~~8~~ 0	7	14		
	(21)	(22)	(10)	(10)		

The maximum penalty is '22' and it occur in the 2nd column and '8' be the smallest cost in this column it occur in the cell (F_3 , W_2). Allocate the largest possible amount in this cell and cross out the 2nd column in the usual manner.

Table – 2

Wear houses / Factory	W_1	W_3	W_4	Factory capacity a_i	
F_1	19 $\boxed{5}$	50	10	~~7~~ 2	(9)
F_2	70	40	60	9	(20)
F_3	40	70	20	10	(20)
Wearhouse Requirement b_j	[illegible]	7	[illegible]		
	[illegible]	[illegible]	[illegible]		

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The maximum penalty is '21' and it occur in the 1st column and '~~19~~' be the smallest cost in this column it occur in the cell (F_1 , W_1). Allocate the largest possible amount in this cell and cross out the 1st column in the usual manner.

Table – 3

Wear houses / Factory	W_3	W_4	Factory capacity a_i	
F_1	50	10	2	(40)
F_2	40	60	9	(20)
F_3	70	20 $\boxed{10}$	~~10~~ 0	(50)
Wearhouse Requirement b_j	7	~~14~~ 4		
	(10)	(10)		

Table – 4

Wear houses / Factory	W_3	W_4	Factory capacity a_i	
F_1	50	10 $\boxed{2}$	~~2~~ 0	(40)
F_2	40	60	9	(20)
Wearhouse Requirement b_j	7	~~4~~ 2		
	(10)	(50)		

Table – 5

Wear houses / Factory	W_3	W_4	Factory capacity a_i	
F_2	40 $\boxed{7}$	60 $\boxed{2}$	~~9~~ ~~7~~ 0	(20)
Wearhouse Requirement b_j	~~7~~ 0	~~2~~ 0		
	(40)	(60)		

= ①

	D_1	D_2	D_3	D_4	Supply
O_1	$\boxed{6}$ 21	$\boxed{5}$ 16	25	13	11
O_2	17	$\boxed{5}$ 18	$\boxed{8}$ 14	23	13
O_3	32	27	$\boxed{4}$ 18	$\boxed{15}$ 41	19
Demand	6	10	12	15	43

Total T. cost $Z = 6\times21 + 5\times16 + 5\times18 + 14\times8 + 18\times4 + 15\times41$

$= 126 + 80 + 90 + 112 + 72 + 615$

$= 1095$

② we use Least-cost method.

	D_1	D_2	D_3	D_4	Supply
O_1	21	16.	25	$\boxed{11}$ 13	11
O_2	$\boxed{1}$ 17	18	$\boxed{12}$ 14	23	13
O_3	$\boxed{5}$ 32	$\boxed{10}$ 27	18	$\boxed{4}$ 41	19
Demand	6	10	12	15	43

Total T. cost $Z = 11\times13 + 1\times17 + 12\times14 + 5\times32 + 10\times27 + 4\times41$

$= 143 + 17 + 168 + 160 + 270 + 164$

T.C $= 922$

⇒ MODI'S (~~modified~~-distributive) METHOD : -

Explain the modi's method for determine the optimum solution of the transportation problem.

Step – 1

Determine an initial basic feasible solution (using one of three methods). Enter this solution in the transportation table.

Step – 2

For an initial basic feasible solution with $m + n - 1$ occupied cells. Calculate u_i and v_i for each row and column.

Step – 3

For all the occupied cells (basic variables x_{ij}) solve the system of equations,

$$C_{ij} = u_i + v_j \text{ ; for all occupied cells (i, j)}$$

Calculate the simplex multipliers starting initially some $u_i = 0$ or $v_j = 0$. It is better to assign zero for a particular u_i or v_j where there are maximum number of allocations in a row or column respectively.

For unoccupied cells (non-basic variables) calculate the opportunity cost d_{ij} by using the relation,

$$d_{ij} = C_{ij} - (u_i + v_j) \text{ ; for all i, j.}$$

Step – 4

Examine the sign of each d_{ij},

- If $d_{ij} > 0$ then the basic feasible solution is optimum.
- If $d_{ij} = 0$ then the basic feasible solution remain unaffected but an alternative solution exists.
- If one or more $d_{ij} < 0$ then an improved solution can be obtained entering unoccupied cell (i, j) in the basis.

An unoccupied cell having the largest negative value of d_{ij} is chosen for entering into the solution.

Step – 5

Construct a loop for the unoccupied cell with largest negative opportunity cost say $x_{ij} = u$ which is constant. Start with this cell and mark a plus (+) sign in the cell. Trace a path along the row or column to an occupied cell, mark the corner with minus (–) sign and continue down the column (or row) to an unoccupied cell and mark the corner with plus and minus sign alternatively. Close the path back to the selected cell.

Step – 6

Select the smallest quantity amongst the cells with minus sign on the corner of the loop. Allocate this value to the selected unoccupied cell and add it to other occupied cells marked with plus sign and subtract it from the occupied cell marked with minus sign.

Step – 7

Obtain a new improved solution by allocating units to the unoccupied cells according to step – 6 and calculate the new transportation cost.

Step – 8

Test the [illegible] further [illegible] all [illegible]

Example : - (1) Determine the optimum basic feasible solution ... using the Modi's Method.

2013

	D_1	D_2	D_3	D_4	Supply
O_1	21	16	25	13	11
O_2	17	18	14	23	13
O_3	32	27	18	41	19
Demand	6	10	12	15	43

→ First we obtain an initial basic feasible solution by Vogel's appro. method.

	D_1	D_2	D_3	D_4	Supply	Row diff
O_1	21	16	25	13 I $\boxed{11}$	~~11~~ 0	(3) X X X X
O_2	17 III $\boxed{6}$	18 VI $\boxed{3}$	14	23 II $\boxed{4}$	~~13~~ ~~9~~, ~~3~~ 0	(3) (3) (3) (4) (18) $\boxed{8}$
O_3	32	27 V $\boxed{7}$	18 IV $\boxed{12}$	41	~~19~~ ~~7~~, 0	(9) (9) (9) $\boxed{(9)}$ $\boxed{(27)}$
Demand	~~6~~ 0,	~~10~~ ~~3~~, 0	~~12~~ 0,	~~15~~ ~~4~~, 0	43	

Column diff

	D_1	D_2	D_3	D_4
	(4)	(2)	(4)	$\boxed{(10)}$
	(15)	(9)	(4)	$\boxed{(18)}$
	$\boxed{(15)}$	(9)	(4)	X
	X	(9)	(4)	X
	X	(9)	X	X
	X	18	X	X

The initial basic feasible solution is shown in the following table.

	D_1	D_2	D_3	D_4	Supply	
O_1	21	16	25	13 $\boxed{11}$	11	$u_1 = -10$
O_2	17 $\boxed{6}$	18 $\boxed{3}$	14	23 $\boxed{4}$	13	$u_2 = 0$
O_3	32	27 $\boxed{7}$	18 $\boxed{12}$	41	19	$u_3 = 9$
Demand	6	10	12	15	43	
	$V_1 = 17$	$V_2 = 18$	$V_3 =$	$V_4 = 23$		

Let u_1, u_2 and u_3 be simplex row multiplies and Let v_1, v_2, v_3 and v_4 be simplex column multiplies.

Take $u_2 = 0$. Max^m allocation in O_2

⇒ For the occupied (basic) cell (i, j),

$C_{ij} = u_i + v_j$

Cell (2, 1); $C_{21} = u_2 + v_1 \Rightarrow 17 = 0 + v_1 \Rightarrow v_1 = 17$

Cell (2, 2); $C_{22} = u_2 + v_2 \Rightarrow 18 = 0 + v_2 \Rightarrow v_2 = 18$

Cell (2, 4); $C_{24} = u_2 + v_4 \Rightarrow 23 = 0 + v_4 \Rightarrow v_4 = 23$

Cell (1, 4); $C_{14} = u_1 + v_4 \Rightarrow 13 = u_1 + 23 \Rightarrow u_1 = -10$

Cell (3, 2); $C_{32} = u_3 + v_2 \Rightarrow 27 = u_3 + 18 \Rightarrow u_3 = 9$

Cell (3, 3); $C_{33} = u_3 + v_3 \Rightarrow 18 = 9 + v_3 \Rightarrow v_3 = 9$

⇒ For the unoccupied (non-basic) cell (i, j) calculate d_{ij} using

$d_{ij} = C_{ij} - (u_i + v_j)$; for all i, j.

$d_{11} = C_{11} - (u_1 + v_1) \Rightarrow d_{11} = 21 - (-10 + 17) = 14$

$d_{12} = C_{12} - (u_1 + v_2) \Rightarrow d_{12} = 16 - (-10 + 18) = 8$

$d_{13} = C_{13} - (u_1 + v_3) \Rightarrow d_{13} = 25 - (-10 + 9) = 26$

$d_{23} = C_{23} - (u_2 + v_3) \Rightarrow d_{23} = 14 - (0 + 9) = 5$

$d_{31} = C_{31} - (u_2 + v_1) \Rightarrow d_{31} = 32 - (9 + 17) = 6$

$d_{34} = C_{34} - (u_3 + v_4) \Rightarrow d_{34} = 41 - (9 + 23) = 9$

For all the unoccupied cells $d_{ij} \geq 0$.

∴ This is the optimum basic feasible solution with minimum cost

$Z = (13 \times 11) + (17 \times 6) + (18 \times 3) + (23 \times 4) + (27 \times 7) + (18 \times 12)$

$= 143 + 102 + 54 + 92 + 189 + 216$

$= 796$.

Example : - (2) Determine the optimum basic feasible solution to the following T.P. using the Modi's Method.(2010)

	D_1	D_2	D_3	D_4	Supply
O_1	7	4	6	5	22
O_2	6	10	3	8	15
O_3	6	8	9	7	8
Demand	7	12	17	9	45

→ First we obtain an initial basic feasible solution by Vogel's appro. method.

	D_1	D_2	D_3	D_4	Supply	
O_1	7	4 $\boxed{12}$	6 $\boxed{2}$	5 $\boxed{8}$	$\cancel{22}$ $\cancel{10}$ $\cancel{8}$ 0	(1)(1)(1)(2) X
O_2	6	10	3 $\boxed{15}$	8	$\cancel{15}$ 0	(3)(3) X X X
O_3	6 $\boxed{7}$	8	9	7 $\boxed{1}$	$\cancel{8}$ $\cancel{1}$ 0	(1)(1)(1)(1)(1
Demand	$\cancel{7}$ 0	$\cancel{12}$ 0	$\cancel{17}$ $\cancel{2}$ 0	$\cancel{9}$ $\cancel{1}$ 0	45	
	(0)	(4)	(3)	(2)		
	(0)	X	(3)	(2)		
	(1)	X	(3)	(2)		
	(1)	X	X	(2)		
	X	X	X	X		

The initial basic feasible solution is shown in the following table.

	D_1	D_2	D_3	D_4	Supply	
O_1	7	4 $\boxed{12}$	6 $\boxed{2}$	5 $\boxed{8}$	$\cancel{22}$ $\cancel{10}$ $\cancel{8}$ 0	$u_1 = 0$
O_2	6	10	3 $\boxed{15}$	8	$\cancel{15}$ 0	$u_2 = -3$
O_3	6 $\boxed{7}$	8	9	7 $\boxed{1}$	$\cancel{8}$ $\cancel{1}$ 0	$u_3 = 2$
Demand	$\cancel{7}$ 0	$\cancel{12}$ 0	$\cancel{17}$ $\cancel{2}$ 0	$\cancel{9}$ $\cancel{1}$ 0	45	
	$v_1 = 4$	$v_2 = 4$	$v_3 = 6$.	$v_4 = 5$		

Let u_1, u_2 and u_3 be simplex row multiplies and Let v_1, v_2, v_3 and v_4 be simplex column multiplies.

Take $u_1 = 0$.

⇒ For the occupied (basic) cell (i, j),

$$C_{ij} = u_i + v_j$$

Cell (1, 2) ; $C_{12} = u_1 + v_2 \Rightarrow 4 = 0 + v_2 \Rightarrow v_2 = 4$

Cell (1, 3) ; $C_{13} = u_1 + v_3 \Rightarrow 6 = 0 + v_3 \Rightarrow v_3 = 6$

Cell (1, 4) : $C_{14} = u_1 + v_4 \Rightarrow 5 = 0 + v_4 \Rightarrow v_4 = 5$

Cell (2, 3) : [illegible] $\Rightarrow 3 = u_2 + 6$ [illegible]

Cell (3, 4) : [illegible] $\Rightarrow 7 = u_3 + 5$ [illegible]

Cell (3, 1) : [illegible] $\Rightarrow 6 = 2$ [illegible]

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$\Rightarrow$ For the unoccupied (non-basic) cell (i, j) calculate d_{ij} using

$$d_{ij} = C_{ij} - (u_i + v_j) ; \text{ for all } i, j.$$

$$d_{11} = C_{11} - (u_1 + v_1) \Rightarrow d_{11} = 7 - (0 + 4) = 3$$

$$d_{21} = C_{21} - (u_2 + v_1) \Rightarrow d_{21} = 6 - (- 3 + 4) = 5$$

$$d_{22} = C_{22} - (u_2 + v_2) \Rightarrow d_{22} = 10 - (- 3 + 4) = 9$$

$$d_{24} = C_{24} - (u_2 + v_4) \Rightarrow d_{24} = 8 - (- 3 + 5) = 6$$

$$d_{32} = C_{32} - (u_3 + v_2) \Rightarrow d_{32} = 8 - (2 + 4) = 2$$

$$d_{33} = C_{33} - (u_3 + v_3) \Rightarrow d_{33} = 9 - (2 + 6) = 1$$

For all the unoccupied cells $d_{ij} \geq 0$.

$\therefore$ This is the optimum basic feasible solution with minimum cost

$$Z = (4 \times 12) + (6 \times 2) + (5 \times 8) + (3 \times 15) + (6 \times 7) + (7 \times 1)$$

$$= 48 + 12 + 40 + 45 + 42 + 7$$

Example : - (4) *Determine the optimum basic feasible solution to the following T.P. using the Modi's Method.*

	W_1	W_2	W_3	W_4	Supply
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Demand	5	8	7	14	34

→ First we obtain an initial basic feasible solution by Vogel's appro. method.

	W_1	W_2	W_3	W_4	Supply	
F_1	19 $\boxed{5}$	30 32	50 60	10 $\boxed{2}$	7	$u_1 = 10$
F_2	70 1	30 − 18	40 $\boxed{7}$	60 $\boxed{2}$	9	$u_2 = 60$
F_3	40 11	8 $\boxed{8}$	70 70	20 $\boxed{10}$	18	$u_3 = 20$
Demand	5	8	7	14	34	
	$v_1 = 9$	$v_2 = -12$	$v_3 = -20$	$v_4 = 0$		

Let u_1, u_2 and u_3 be simplex row multiplies and Let v_1, v_2 , v_3 and v_4 be simplex column multiplies.

Take $v_4 = 0$.

⇒ For the occupied (basic) cell (i, j),

$$C_{ij} = u_i + v_j$$

Cell (1, 4) ; $C_{14} = u_1 + v_4 \Rightarrow 10 = u_1 + 0 \Rightarrow u_1 = 10$

Cell (1, 1) ; $C_{11} = u_1 + v_1 \Rightarrow 19 = 10 + v_1 \Rightarrow v_1 = 9$

Cell (2, 4) ; $C_{24} = u_2 + v_4 \Rightarrow 60 = u_2 + 0 \Rightarrow u_2 = 60$

Cell (2, 3) ; $C_{23} = u_2 + v_3 \Rightarrow 40 = 60 + v_3 \Rightarrow v_3 = -20$

Cell (3, 4) ; $C_{34} = u_3 + v_4 \Rightarrow 20 = u_3 + 0 \Rightarrow u_3 = 20$

Cell (3, 2) ; $C_{32} = u_3 + v_2 \Rightarrow 8 = 20 + v_2 \Rightarrow v_2 = -12$

⇒ For the unoccupied (non-basic) cell (i, j) calculate d_{ij} using

$d_{ij} = C_{ij} - (u_i + v_j)$: for all i , j.

$d_{12} = C_{12} - (u_1 + v_2)$ ⇒ d_{12} = 30 − (10 − 12) = 32

$d_{13} = C_{13} - (u_1 + v_3)$ [illegible]

$d_{21} = C_{21} - (u_2 + v_1) \Rightarrow d_{21} = 70 - (60 + 9) = 1$

$d_{22} = C_{22} - (u_2 + v_2) \Rightarrow d_{22} = 30 - (60 - 12) = -18$

$d_{31} = C_{31} - (u_3 + v_1) \Rightarrow d_{31} = 40 - (20 + 9) = 11$

$d_{33} = C_{33} - (u_3 + v_3) \Rightarrow d_{33} = 70 - (20 - 20) = 70$

Here,

$d_{22} = -18$ is the most negative variable in the cell (F_2, W_2) and so i not the basic optimum solution.

We identify the loop starting and ending with this cell (F_2, W_2) An mark a plus sign '+' in this cell, trace a path along the row to an occupi cell and mark the corner with minus sign '–' and continue down the co to an occupied cell and mark the corner with plus sign '+' and minus si alternatively. Close the path back to the selected unoccupied cell.

Min { 8, 2 } = 2

$\Rightarrow x_{22} = 2 : x_{32} = 8 - 2 = 6 : x_{24} = 2 - 2 = 0$

and $x_{34} = 10 + 2 = 12$

	W_1	W_2	W_3	W_4	Supply
F_1	19 $\boxed{5}$	30	50	10 $\boxed{2}$	7
F_2	70	30 (+) −18	40 $\boxed{7}$	60 (−) $\boxed{2}$	9
F_3	40	8 (−) $\boxed{8}$	70	20 (+) $\boxed{10}$	18
Demand	5	8	7	14	34

The basic variable x_{24} will leave the basis and x_{22} will enter into the basi

	W_1	W_2	W_3	W_4	Supply	
F_1	19 $\boxed{5}$	30 32	50 42	10 $\boxed{2}$	7	$u_1 = 0$
F_2	70 19	30 $\boxed{2}$	40 $\boxed{7}$	60	9	$u_2 = 3$
F_3	40 11	8 $\boxed{6}$	70 70	20 $\boxed{12}$	18	$u_3 = 1$
Demand	5	8	7	14	34	

$V_1 = 10$ $V_2 = -2$ $V_3 = 8$ $V_4 = 10$

Take $u_1 = 0$.

$\Rightarrow$ For the occupied cell (i, j),

$$u_i + v_j = C_{ij}$$

Cell (1, 1); $C_{11} = u_1 + v_1 \Rightarrow 19 = 0 + v_1 \Rightarrow v_1 = 19$

Cell (1, 4); $C_{14} = u_1 + v_4 \Rightarrow 10 = 0 + v_4 \Rightarrow v_4 = 10$

Cell (3, 4); $C_{34} = u_3 + v_4 \Rightarrow 20 = u_3 + 10 \Rightarrow u_3 = 10$

Cell (3, 2); $C_{32} = u_3 + v_2 \Rightarrow 8 = 10 + v_2 \Rightarrow v_2 = -2$

Cell (2, 2); $C_{22} = u_2 + v_2 \Rightarrow 30 = u_2 - 2 \Rightarrow u_2 = 32$

Cell (2, 3); $C_{23} = u_2 + v_3 \Rightarrow 40 = 32 + v_3 \Rightarrow v_3 = 8$

$\Rightarrow$ For the unoccupied cell (i, j) calculate d_{ij} using

$$d_{ij} = C_{ij} - (u_i + v_j); \text{ for all } i, j.$$

$d_{12} = C_{12} - (u_1 + v_2) \Rightarrow d_{12} = 30 - (0 - 2) = 32$

$d_{13} = C_{13} - (u_1 + v_3) \Rightarrow d_{13} = 50 - (0 + 8) = 42.$

$d_{21} = C_{21} - (u_2 + v_1) \Rightarrow d_{21} = 70 - (32 + 19) = 19$

$d_{24} = C_{24} - (u_2 + v_4) \Rightarrow d_{24} = 60 - (32 + 10) = 18$

$d_{31} = C_{31} - (u_3 + v_1) \Rightarrow d_{31} = 40 - (10 + 19) = 11$

$d_{33} = C_{33} - (u_3 + v_3) \Rightarrow d_{33} = 70 - (10 + 8) = 52$

For all the unoccupied cells $d_{ij} \geq 0$.

$\therefore$ This is the optimum basic feasible solution.

$\therefore$ Total cost $Z = (19 \times 5) + (10 \times 2) + (30 \times 2) + (40 \times 7) + (8 \times 6) + (20 \times 12)$

$= 95 + 20 + 60 + 280 + 48 + 240$

$= 743.$

Example : - (5) Determine the optimum basic feasible solution to the following T.P. using the Modi's Method.(2010)

	P	Q	R	S	Supply
A	6	3	5	4	22
B	5	9	2	7	15
C	5	7	8	6	8
Demand	7	12	17	9	45

$\Rightarrow$ First we obtain an initial basic feasible solution by Vogel's approximation method.

Hungarian Method

Example : - Five persons A, B, C, D and E are to assigned to five jobs 1, 2, 3, 4 and 5. The cost matrix is given as under.

1

Persons ↓ \ Jobs →	1	2	3	4	5
A	8	4	2	6	1
B	0	9	5	5	4
C	3	8	9	2	6
D	4	3	1	0	3
E	9	5	8	9	5

Find the proper assignment (the optimum assignment).

⇒ Locate the smallest number (element) in each row of the given cost table and then subtract that from each number of that row. We get,

Take smallest in each row & subtract

Persons ↓ \ Jobs →	1	2	3	4	5
A	7	3	1	5	0
B	0	9	5	5	4
C	1	6	7	0	4
D	4	3	1	0	3
E	4	0	3	4	0

⇒ Locate the smallest number (element) in each column of the given cost table and then subtract that from each number of that column. We get,

Take smallest in each column & subtract

Persons ↓ \ Jobs →	1	2	3	4	5
A	7	3	0	5	0
B	0	9	4	5	4
C	1	6	6	0	4
D	4	3	0	0	3
E	4	0	2	4	0

⇒ Covering the zeroes with the minimum number of horizontal and vertical lines. We have,

Jobs → / Persons ↓	1	2	3	4	5
A	7	3	0	5	0 ✓
B	0 ✓	9	4	5	4
C	1	6	6	0 ✓	4
D	4	3	0 ✓	0	3
E	4	0 ✓	2	4	0

Here,

the minimum number of lines covering zeros = the order of cost matrix

∴ The problem is ready for assignment.

Now mark the assigned zero by tick (✓) sign.

Here,

the numbers of assigned cells is equal to the number of rows or columns so it is an optimal solution.

The total cost associated with this solution is obtained by adding original cost values in the occupied cells.

Solution	cost value
A → 5	1
B → 1	0
C → 4	2
D → 3	1
E → 2	5

∴ The cost $Z = 1 + 0 + 2 + 1 + 5$

$Z = 9$

Example : - A department has five employes A, B, C, D and E with five jobs 1, 2, 3, 4 and 5 to performed. The time (in hours) that each employee Will take to perform a job is as under. How should the jobs be allotted one employee so as minimize the total man hours ? each job is given ... the effectiveness ...

Jobs → / Employee ↓	1	2	3	4	5
A	10	5	13	15	16
B	3	9	18	13	6
C	10	7	2	2	2
D	7	11	9	7	12
E	7	9	10	4	12

Row min
5, 3, 2, 4

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⇒ Locate the smallest number (element) in each row of the given cost table and then subtract that from each number of that row. We get,

Jobs →
Employee ↓

	1	2	3	4	5
A	5	0	8	10	11
B	0	6	15	10	3
C	8	5	0	0	0
D	0	4	2	0	5
E	3	5	6	0	8

Column min
0,0,0,0,0

⇒ Locate the smallest number (element) in each column of the given cost table and then subtract that from each number of that column. We get,

Jobs →
Employee ↓

	1	2	3	4	5
A	5	0	8	10	11
B	0	6	15	10	3
C	8	5	0	0	0
D	0	4	2	0	5
E	3	5	6	0	8

⇒ Covering the zeroes with the minimum number of horizontal and vertical lines. We have,

Jobs →
Employee ↓

	1	2	3	4	5
A	5	0	8	10	11
B	0	6	15	10	3
C	8	5	0	0	0
D	0	4	2	0	5
E	3	5	6	0	8

remain number = 2

Here,
the minimum number of lines covering zeros = 4
and the order of cost matrix = 5

∴ The problem is not ready for assignment. The minimum uncoverd element is '2'.

Now subtract '2' from all of the uncovered numbers. And add it to those numbers that are at the intersection of horizontal and vertical lines. We have,

Employee ↓ / Jobs →	1	2	3	4	5
A	5	0 ✓	6	10	9
B	0 ✓	6	13	10	1
C	10	7	0	2	0 ✓
D	0	1	0 ✓	0	3
E	3	5	4	0 ✓	6

Now covering the zeroes with the minimum number of horizontal and vertical lines. We have,

the minimum number of lines covering zeros = the order of cost matrix

∴ The problem is ready for assignment.

Now mark the assigned zero by tick (✓) sign.

Here,

the numbers of assigned cells is equal to the number of rows or columns so it is an optimal solution.

The total cost associated with this solution is obtained by adding original cost values in the occupied cells.

Solution	cost value
A → 2	5
B → 1	3
C → 5	2
D → 3	9
E → 4	4

∴ The cost $Z = 5 + 3 + 2 + 9 + 4$
$= 23.$

Example : - Solve the following 4 × 4 assignment problem to maximize the total production.

Remark:- If we have given a maximization problem then convert it into minimization problem, simply, multiplying by -1 to each entry in the effectiveness matrix and then solve it in the usual manner.

Man ↓ / Machine →	a	b	c	d
1	40	45	42	39
2	38	45	46	40
3	42	46	46	43
4	41	44	42	44

Convert the problem into minimization problem multiplying by – 1 to each entry in the given matrix.

Machine → Man ↓	a	b	c	d
1	– 40	– 45	– 42	– 39
2	– 38	– 45	– 46	– 40
3	– 42	– 46	– 46	– 43
4	– 41	– 44	– 42	– 44

⇒ Locate the smallest number (element) in each row of the given cost table and then subtract that from each number of that row. We get

Machine → Man ↓	a	b	c	d
1	5	0	3	6
2	8	1	0	6
3	4	0	0	3
4	3	0	2	0

⇒ Locate the smallest number (element) in each column of the given cost table and then subtract that from each number of that column. We get,

Machine → Man ↓	a	b	c	d
1	2	0	3	6
2	5	1	0	6
3	1	0	0	3
4	0	0	2	0

⇒ Covering the zeroes with the minimum number of horizontal and vertical lines. We have,

Man ↓ \ Machine →	a	b	c	d
1	2	0	3	6
2	5	1	0	6
3	1	0	0	3
4	0	0	2	0

R+C
2+2=4
4=4 optimal
min=2

Here,
the minimum number of lines covering zeros = 3
And the order of cost matrix = 4
∴ The problem is not ready for assignment. The minimum uncoverd element is '1'.

Now subtract '1' from all of the uncovered numbers. And add it to those numbers that are at the intersection of horizontal and vertical lines. We have,

Man ↓ \ Machine →	a	b	c	d
1	1	0 ✓	3	5
2	4	1	0 ✓	5
3	0 ✓	0	0	2
4	0	1	3	0 ✓

	a	b	c	d
1	0	$\boxed{0}$	3	4
2	3	1	$\boxed{0}$	4
3	1	2	2	3
4	$\boxed{0}$	2	4	$\boxed{0}$

Now covering the zeroes with the minimum number of horizontal and vertical lines. We have,

the minimum number of lines covering zeros
= the order of cost matrix

∴ The problem is ready for assignment.

Now mark the assigned zero by tick (✓) sign.

Here,

the numbers of assigned cells is equal to the number of rows or columns so it is an optimal solution.

The total cost associated with this solution is obtained by adding original cost values in the occupied cells.

Solution	cost value
1 → b	45
2 → c	46
3 → a	42
4 → d	44

∴ The cost Z = 45 + 46 + 42 + 44
= 177.

Example : - Using the following cost matrix, solve the assignment problem to minimize total cost. (2009).

	I	II	III	IV	V
A	10	3	3	2	8
B	9	7	8	2	7
C	7	5	6	2	4
D	3	5	8	2	4
E	9	10	9	6	10

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Example : - Given the following time matrix (in hours), solve the assignment problem by enumeration method to find the minimum total time of assignment. (2009).

	I	II	III
A	120	100	80
B	80	90	110
C	110	140	120

280

Example : Determine the optimum basic feasible solution to the following T.P. using the Modi's Method.

	D_1	D_2	D_3	D_4	D_5	D_6	Supply
O_1	9	12	9	6	9	10	5
O_2	7	3	7	7	5	5	6
O_3	6	5	9	11	3	11	2
O_4	6	8	11	2	2	10	9
Demand	4	4	6	2	4	2	22

→ First we obtain an initial basic feasible solution by Vogel's appro. method.

	D_1	D_2	D_3	D_4	D_5	D_6	a_i	
O_1	9	12	9 $\boxed{5}$	6	9	10	~~5~~ 0	(3)(3)(0)(0)(0)(0)
O_2	7	3 $\boxed{4}$	7	7	5	5 $\boxed{2}$	~~6~~ ~~4~~ 0	(2)(2)(2)(**4**) X X
O_3	6 $\boxed{1}$	5	9 $\boxed{1}$	11	3	11	~~2~~ ~~1~~ 0	(2)(2)(2)(1)(1)(**3**)
O_4	6 $\boxed{3}$	8	11	2 $\boxed{2}$	2 $\boxed{4}$	10	~~9~~ ~~7~~ ~~3~~ 0	(0)(0)(**4**)(2)(2)(**5**)
b_j	~~4~~ ~~1~~ 0	~~4~~ 0	~~6~~ ~~1~~ 0	~~2~~ 0	~~4~~ 0	~~2~~ 0	22	
	(0)	(2)	(2)	(4)	(1)	(**5**)		
	(0)	(2)	(2)	(**4**)	(1)	X		
	(0)	(2)	(2)	X	(1)	X		
	(0)	(2)	(2)	X	X	X		
	(0)	X	(0)	X	X	X		

The initial basic feasible solution is shown in the following table.

	D_1	D_2	D_3	D_4	D_5	D_6	a_i
O_1	9	12	9 $\boxed{5}$	6	9	10	5
O_2	7	3 $\boxed{4}$	7	7	5	5 $\boxed{2}$	6
O_3	6 $\boxed{1}$	5	9 $\boxed{1}$	11	3	11	2
O_4	6 $\boxed{3}$	8	11	2 $\boxed{2}$	2 $\boxed{4}$	10	9
b_j	4	4	6	2	4	2	22

Here, $m = 4$, $n = 6 \Rightarrow m + n - 1 = 4 + 6 - 1 = 9$;

the number of positive allocations = 8 < 9

$\therefore$ The solution is degenerate solution. In order to remove degeneracy we assign 'ε' to unoccupied cell (O_2, D_3) as shown in the below table.

	D_1	D_2	D_3	D_4	D_5	D_6	a_i	
O_1	9	12	9 (5)	6	9	10	**5**	$u_1 = 9$
O_2	7	3 (4)	7 (ε)	7	5	5 (2)	**6**	$u_2 = 7$
O_3	6 (1)	5	9 (1)	11	3	11	**2**	$u_3 = 9$
O_4	6 (3)	8	11	2 (2)	2 (4)	10	**9**	$u_4 = 9$
b_j	4	4	6	2	4	2	**22**	
	$v_1 = -3$	$v_2 = -4$	$v_3 = 0$	$v_4 = -7$	$v_5 = -7$	$v_6 = -2$		

Let u_1, u_2, u_3 and u_4 be simplex row multiplies and Let v_1, v_2 v_3, v_4, v_5 and v_6 be simplex column multiplies.

Take $v_3 = 0$.

$\Rightarrow$ For the occupied (basic) cell (i, j),

$$C_{ij} = u_i + v_j$$

Cell (1, 3); $C_{13} = u_1 + v_3 \Rightarrow 9 = u_1 + 0 \Rightarrow u_1 = 9$

Cell (2, 3); $C_{23} = u_2 + v_3 \Rightarrow 7 = u_2 + 0 \Rightarrow u_2 = 7$

Cell (3, 3); $C_{33} = u_3 + v_3 \Rightarrow 9 = u_3 + 0 \Rightarrow u_3 = 9$

Cell (2, 2); $C_{22} = u_2 + v_2 \Rightarrow 3 = 7 + v_2 \Rightarrow v_2 = -4$

Cell (2, 6); $C_{26} = u_2 + v_6 \Rightarrow 5 = 7 + v_6 \Rightarrow v_6 = -2$

Cell (3, 1); $C_{31} = u_3 + v_1 \Rightarrow 6 = 9 + v_1 \Rightarrow v_1 = -3$

Cell (4, 1); $C_{41} = u_4 - 3 \Rightarrow 6 = u_4 - 3 \Rightarrow u_4 = 9$

Cell (4, 4); $C_{44} = u_4 + v_4 \Rightarrow 2 = 9 + v_4 \Rightarrow v_4 = -7$

Cell (4, 5); $C_{45} = u_4 + v_5 \Rightarrow 2 = 9 + v_5 \Rightarrow v_5 = -7$

$\Rightarrow$ For the unoccupied (non-basic) cell (i, j) calculate d_{ij} using

$d_{ij} = C_{ij} - (u_i + v_j)$; for all i, j.

$d_{11} = C_{11} - (u_1 + v_1) \quad \Rightarrow d_{11} = 9 - (9 - 3) = 3$

$d_{12} = C_{12} - (u_1 + v_2) \quad \Rightarrow d_{12} = 12 - (9 - 4) = 7$

$d_{14} = C_{14} - (u_1 + v_4) \quad \Rightarrow d_{14} = 6 - (9 - 7) = 4$

$d_{15} = C_{15} - (u_1 + v_5) \quad \Rightarrow d_{15} = 9 - (9 - 7) = 7$

$d_{16} = C_{16} - (u_1 + v_6) \quad \Rightarrow d_{16} = 10 - (9 - 2) = 3$

$d_{21} = C_{21} - (u_2 + v_1) \quad \Rightarrow d_{21} = 7 - (7 - 3) = 3$

$d_{24} = C_{24} \quad (u_2 + v_4) \quad \Rightarrow d_{24} = 7 - (7 - 7) - 7$

$d_{25} = C_{25} - (u_2 + v_5) \quad \Rightarrow d_{25} = 5 - (7 - 7) = 5$

$d_{32} = C_{32} - (u_2 + v_2) \quad \Rightarrow d_{32} = 5 - (9 - 4) = 0$

$d_{34} = C_{34} - (u_3 + v_4) \quad \Rightarrow d_{34} = 11 - (9 - 7) = 9$

$d_{35} = C_{35} - (u_3 + v_5) \quad \Rightarrow d_{35} = 3 - (9 - 7) = 1$

$d_{36} = C_{36} - (u_3 + v_6) \quad \Rightarrow d_{36} = 11 - (9 - 2) = 4$

$d_{42} = C_{42} - (u_4 + v_2) \quad \Rightarrow d_{42} = 8 - (9 - 4) = 3$

4

$d_{43} = C_{43} - (u_4 + v_3) \Rightarrow d_{43} = 11 - (9 - 0) = 2$

$d_{46} = C_{46} - (u_4 + v_6) \Rightarrow d_{46} = 10 - (9 - 2) = 3$

For all the unoccupied cells $d_{ij} \geq 0$.

$\therefore$ This is the optimum basic feasible solution with minimum cost z,

$$= (9 \times 5) + (3 \times 4) + (7 \times \varepsilon) + (5 \times 2) + (6 \times 1) + (9 \times 1) + (6 \times 3) + (2 \times 2) + (2 \times 4)$$

$$= 45 + 12 + 7\varepsilon + 10 + 6 + 9 + 18 + 4 + 8$$

$$= 112 + 7\varepsilon. \quad \text{As } \varepsilon \to 0$$

$$= 112.$$

The differences between assignment problem and transportation problem.

1. TP has supply and demand constraints while AP does not have the same.
2. The optimal test for TP is when all cell evaluations are greater than or equal to zero whereas in AP the number of lines must be equal to the size of matrix.
3. A TP sum is balanced when demand is equal to supply, and an AP sum is balanced when number of rows are equal to the number of columns. (Total supply must equal to total demand in the transportation problem, but each supply and demand value is 1 in the assignment problem.)
4. for AP. We use Hungarian method and for transportation we use MODI Method
5. In AP. We have to assign different jobs to different entities while in TP we have to find optimum transportation cost.

⇒ Triangular ...

MAT-304 (Mathematical programming and operation Research.

1. Convex Analysis

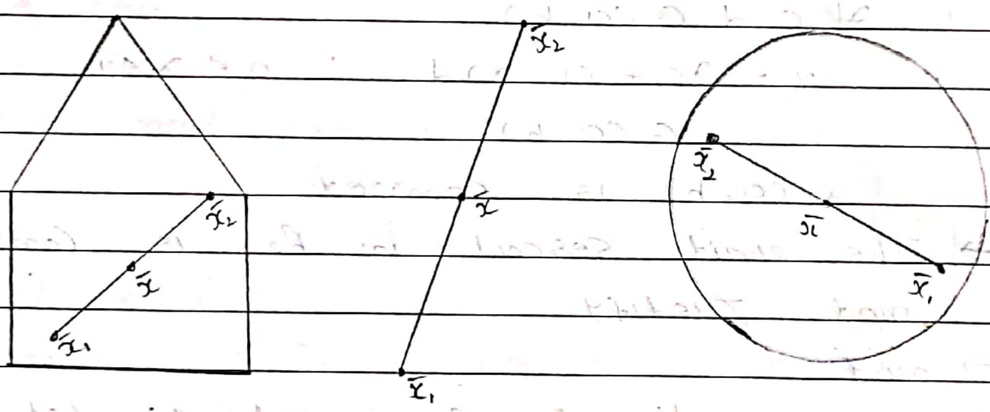
Unit-01

* **Convex linear Combination:-** Let $\bar{x}_1, \bar{x}_2 \in E$ then the pt $\bar{x} = \lambda\bar{x}_1 + (1-\lambda)\bar{x}_2$ / $\bar{x} = (1-\lambda)\bar{x}_1 + \lambda\bar{x}_2$ where $0 \le \lambda \le 1$ is called linear combination of $\bar{x}_1, \bar{x}_2$ that is $\bar{x}$ is any pt lying on the line segment joining x_1 & x_2

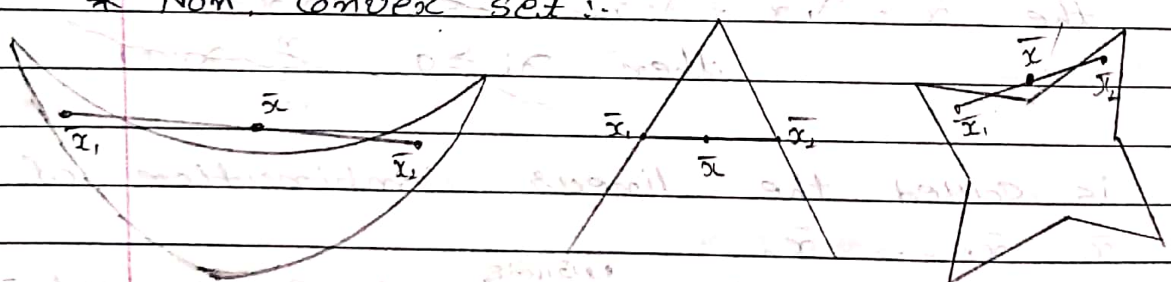
* **Convex Set:-** A set $K \subset E_n$ / E^n is said to be convex set if the convex linear combination of any two pts in K is belongs to K.

i.e. K is convex set if $\bar{x} \in K$ where $\bar{x} = \lambda\bar{x}_1 + (1-\lambda)\bar{x}_2$, $0 \le \lambda \le 1$, $\bar{x}_1, \bar{x}_2 \in K$.

* Convex set:-



* Non Convex set:-



As, $\bar{x}_1, \bar{x}_2 \in K$ but $\bar{x} \notin K$

As, $\bar{x}_1, \bar{x}_2 \in S$ but $\bar{x} \notin S$

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Ex Any connected interval $[a,b]$, $[a,b)$, $(a,b]$ or (a,b) in $\mathbb{R}$ is always convex set.

Solⁿ: (a,b) is convex.

let $c, d \in (a,b)$ with $c<d$

Consider $x = \lambda c + (1-\lambda) d$, $0 \le \lambda \le 1$

then show that $x \in (a,b)$

$a < c = \lambda c + (1-\lambda) c$, $0 \le \lambda \le 1$

$a < \lambda c + (1-\lambda) c$

$< \lambda c + (1-\lambda) d$

$= x$

$< \lambda c + (1-\lambda) d$ $\quad \because c<d$

$= d$

$< b$ $\quad \because d \in (a,b)$

$\forall c, d \in (a,b)$

$x = \lambda c + (1-\lambda) d$; $0 \le \lambda \le 1$

$\in (a,b)$

$\therefore (a,b)$ is convex.

→ The unit sercal in R_2 is convex or not Justify.

→ not

* Convex linear Combination:- let $\bar{x}_i \in E_n$; $i = 1, 2 \dots n$

the $\bar{x} = \lambda_1 \bar{x}_1 + \lambda_2 \bar{x}_2 + \dots + \lambda_n \bar{x}_n$

then $\lambda_i \ge 0$, $\sum_{i=1}^{n} \lambda_i = 1$

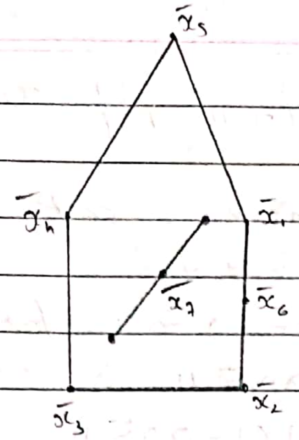
is called the linear combination of $\bar{x}_1, \bar{x}_2 \dots \bar{x}_n$

* Extreme Point (vertex) शिरोबिंदु of convex set:-

A point x of a convex set k is a vertex of convex set k is a vertex of a convex set k if it is not possible to find pts $\bar{x}_1, \bar{x}_2 \in k$ such that $\bar{x} = \lambda \bar{x}_1 + (1-\lambda) \bar{x}_2$

where

Vector ‖ ‖ - लाभ.
Scalar | | - मॉड



Here $\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5$ are extreme pt but x_6 and x_7 are not extreme pt.

Remark (i) Here $0 < \lambda < 1$: the vertex does not lies on a line segment joining to other pts in set K.

(ii) the vertex is the boundary pt of are set but all the boundary pts are not a vertex of a set.

Ex the unit ball in a Euclidean space E_n is convex.

Solⁿ: the unit ball

$$B = \{ \bar{x} \in E_n \mid \|\bar{x}\| \le 1 \}$$

let $\bar{x}, \bar{y} \in B$

$\Rightarrow \bar{x}, \bar{y} \in E_n$ & $\|\bar{x}\| \le 1$, $\|\bar{y}\| \le 1$

Consider $\bar{\omega} = \lambda \bar{x} + (1-\lambda)\bar{y}$ for some $0 \le \lambda \le 1$

Now, $\bar{\omega} = \lambda\bar{x} + (1-\lambda)\bar{y} \in E_n$

and,
$$\begin{aligned}
\|\bar{\omega}\| &= \|\lambda\bar{x} + (1-\lambda)\bar{y}\| \\
&\le \|\lambda\bar{x}\| + \|(1-\lambda)\bar{y}\| \\
&= |\lambda| \, \|\bar{x}\| + |1-\lambda| \, \|\bar{y}\| \\
&\le |\lambda| + |1-\lambda| \quad \because \|\bar{x}\| \le 1, \ \|\bar{y}\| \le 1 \\
&= \lambda + 1 - \lambda \\
&= 1
\end{aligned}$$

$\left\{ \begin{array}{l} \text{As } 0 \le \lambda \le 1 \\ \Rightarrow 0 \le 1-\lambda \le 1 \end{array} \right.$

$\therefore \|\bar{\omega}\| \le 1$

$\therefore \bar{\omega} \in B$

$\forall \ \bar{x}, \bar{y} \in B$

$\bar{\omega} = \lambda\bar{x} + (1-\lambda)\bar{y} \in B$, for $0 \le \lambda \le 1$

Then unit ... conv...

PAGE
DATE

Ex So that the set $S = \{(x_1, x_2) \mid 3x_1^2 + 2x_2^2 \le 6\}$ in a convex set.

Sol^n: Let $\bar{x}, \bar{y} \in S$ with

$\bar{x} = (x_1, x_2)$, $\bar{y} = (y_1, y_2)$

Consider $\bar{\omega} = \lambda\bar{x} + (1-\lambda)\bar{y}$, for some $0 \le \lambda \le 1$

If $\bar{\omega} = (\omega_1, \omega_2)$ then

$\omega_1 = \lambda x_1 + (1-\lambda) y_1$

$\omega_2 = \lambda x_2 + (1-\lambda) y_2$

Now, $3\omega_1^2 + 2\omega_2^2$

$= 3(\lambda x_1 + (1-\lambda)y_1)^2 + 2(\lambda x_2 + (1-\lambda)y_2)^2$

$= \lambda^2 (3x_1^2 + 2x_2^2) + (1-\lambda)^2 (3y_1^2 + 2y_2^2)$

$+ 2\lambda(1-\lambda)(3x_1y_1 + 2x_2y_2)$

Here, $3x_1y_1 + 2x_2y_2 =$

$= \sqrt{(3x_1y_1 + 2x_2y_2)^2}$

$= \sqrt{9x_1^2y_1^2 + 4x_2^2y_2^2 + 12x_1y_1x_2y_2}$

$= \sqrt{9x_1^2y_1^2 + 4x_2^2y_2^2 + 6(x_2^2y_2^2 + x_2^2y_1^2)}$

$= \sqrt{(\sqrt{3}x_1)^2 + (\sqrt{2}x_2)^2} \sqrt{(\sqrt{3}y_1)^2 + (\sqrt{2}y_2)^2}$

$\le \sqrt{6}\sqrt{6}$

$= 6$

$\le \lambda^2(6) + (1-\lambda)^2(6) + 2\lambda(1-\lambda)(6)$

$= 6(\lambda^2 + 1 - \lambda - \lambda + \lambda^2 + 2\lambda - 2\lambda^2)$

$= 6$

$3\omega_1^2 + 2\omega_2^2$

$\bar{\omega} \in S$

$\forall\ \bar{x}, \bar{y} \in S$

$\bar{\omega} = \lambda\bar{x} + (1-\lambda)\bar{y} \in S$

where $0 \le \lambda \le 1$

$\therefore$ S is convex set.

Ex define a convex set. Prove that the set $S = \{x \mid x \in E_n \mid \|\bar{x}\| \le 1\}$ is a convex subset of Euclidean space E also give convex subset of E.

Sol^n: the unit set,

$$S = \{ \bar{x} \in E_2 \mid \|\bar{x}\| \le 1 \}$$

Let $\bar{x}, \bar{y} \in S$.

$\Rightarrow \bar{x}, \bar{y} \in E_2$ & $\|\bar{x}\| \le 1$, $\|\bar{y}\| \le 1$

Consider $\bar{\omega} = \lambda \bar{x} + (1-\lambda)\bar{y}$ for some $0 \le \lambda \le 1$

Now, $\bar{\omega} = \lambda \bar{x} + (1-\lambda)\bar{y} \in E_2$

and

$$\|\bar{\omega}\| = \|\lambda \bar{x} + (1-\lambda)\bar{y}\|$$
$$\le \|\lambda \bar{x}\| + \|(1-\lambda)\bar{y}\|$$
$$\le |\lambda| \, \|\bar{x}\| + |1-\lambda| \, \|\bar{y}\|$$
$$\le |\lambda| + |1-\lambda| \qquad \because \|\bar{x}\| \le 1, \ \|\bar{y}\| \le 1$$
$$= \lambda + 1 - \lambda$$
$$= 1$$

$\|\bar{\omega}\| \le 1$

$\bar{\omega} \in S$

$\forall \ \bar{x}, \bar{y} \in S$

$\bar{\omega} = \lambda \bar{x} + (1-\lambda)\bar{y} \in S$, for $0 \le \lambda \le 1$

The unit set is convex.

$\Rightarrow$ Consider,

$$S = \{ \bar{x} \in E_2 \mid \|\bar{x}\| = 1 \}$$

Here S is not a convex set.

Let $\bar{x}, \bar{y} \in S$ with $\bar{x} \ne \bar{y}$

Consider, $\bar{\omega} = \lambda \bar{x} + (1-\lambda)\bar{y}$ for $0 \le \lambda \le 1$

Now, $\|\bar{\omega}\| = \|\lambda \bar{x} + (1-\lambda)\bar{y}\|$

$\le \|\lambda \bar{x}\| + \|(1-\lambda)\bar{y}\|$

for $0 < \lambda < 1$

equality possible iff (1) $\bar{x} = 0$, (2) $\bar{y} = 0$

(3) $\lambda \bar{x} = t(1-\lambda)\bar{y}$ for $t > 0$

since $\bar{x}, \bar{y} \in S$

$\Rightarrow \|\bar{x}\| = 1 = \|\bar{y}\|$

$\Rightarrow \bar{x} \ne 0$, $\bar{y} \ne 0$.

$\Rightarrow$ Neither (i) nor (ii) is true.
if (iii) is satisfied i.e. $\lambda\bar{x} + (1-\lambda)\bar{y}$, $\bar{x}$ sorts
then It is necessary that
$||\lambda\bar{x}|| = ||t(1-\lambda)\bar{y}||$
$\Rightarrow |\lambda| \, ||\bar{x}|| = |t(1-\lambda)| \, ||\bar{y}||$
$\Rightarrow \lambda = t(1-\lambda)$, $||\bar{x}|| = 1 = ||\bar{y}||$
$\Rightarrow \bar{x} = \bar{y}$ & $0 < \lambda < 1$, $t > 0$
which contradicts the fact that $\bar{x} \neq \bar{y}$.

$\therefore ||\bar{\omega}|| < ||\lambda\bar{x}|| + ||(1-\lambda)\bar{y}||$
$\leq |\lambda| \, ||\bar{x}|| + |1-\lambda| \, ||\bar{y}||$
$= \lambda + 1 - \lambda$
$= 1$

$||\bar{\omega}|| < 1$
$\bar{\omega} \notin S$
$\therefore S$ is not convex set.

EX let S_1 and S_2 be two convex sets in E_n and $\alpha \in \mathbb{R}$ then show that
(1) $\alpha S_1 = \{\bar{z} \in E_n \mid \bar{z} = \alpha\bar{x}, \bar{x} \in S_1\}$
(2) $S_1 + S_2 = \{\bar{z} \in E_n \mid \bar{z} = \bar{x} + \bar{y}, \bar{x} \in S_1, \bar{y} \in S_2\}$
are convex sets.

Solⁿ: (1) $\alpha S_1 = \{\bar{z} \in E_n \mid \bar{z} = \alpha\bar{x}, \bar{x} \in S_1\}$
let $\bar{z}_1, \bar{z}_2 \in \alpha S_1$
$\bar{z}_1 = \alpha\bar{x}_1$, $\bar{z}_2 = \alpha\bar{x}_2$
for some $\bar{x}_1, \bar{x}_2 \in S_1$
Now, $\lambda\bar{z}_1 + (1-\lambda)\bar{z}_2$ for $0 \leq \lambda \leq 1$
$= \lambda(\alpha\bar{x}_1) + (1-\lambda)(\alpha\bar{x}_2)$
$= \alpha(\lambda\bar{x}_1 + (1-\lambda)\bar{x}_2)$
$\in \alpha S_1$
As $\bar{x}_1, \bar{x}_2 \in S_1$ and S_1 is convex
$\therefore$ for $0 \leq \lambda \leq 1$
$\lambda\bar{x}_1 + (1-\lambda)\bar{x}_2 \in S_1$

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$\therefore \forall\ \bar{z}_1, \bar{z}_2 \in \alpha S_1$

$\lambda \bar{z}_1 + (1-\lambda)\bar{z}_2 \in \alpha S_1$

$\therefore \alpha S_1$ is a convex set.

(2) $S_1 + S_2 = \{ \bar{z} \in E_n \mid \bar{z} = \bar{x} + \bar{y},\ \bar{x} \in S_1, \bar{y} \in S_2 \}$

let $\bar{z}_1, \bar{z}_2 \in S_1 + S_2$

$\Rightarrow \bar{z}_1 = \bar{x}_1 + \bar{y}_1, \quad \bar{z}_2 = \bar{x}_2 + \bar{y}_2$

$\bar{x}_1, \bar{x}_2 \in S_1, \ \bar{y}_1, \bar{y}_2 \in S_2$

Now, $\lambda \bar{z}_1 + (1-\lambda)\bar{z}_2$ for $0 \le \lambda \le 1$

$= \lambda(\bar{x}_1 + \bar{y}_1) + (1-\lambda)(\bar{x}_2 + \bar{y}_2)$

$= (\lambda \bar{x}_1 + (1-\lambda)\bar{x}_2) + (\lambda \bar{y}_1 + (1-\lambda)\bar{y}_2) \in S$

As $\bar{x}_1, \bar{x}_2 \in S_1$ & S_1 is convex.

$\Rightarrow \lambda \bar{x}_1 + (1-\lambda)\bar{x}_2 \in S_1$

Wly $\bar{y}_1, \bar{y}_2 \in S_2$ & S_2 is convex

$\Rightarrow \lambda \bar{y}_1 + (1-\lambda)\bar{y}_2 \in S_2$

$\therefore \forall\ \bar{z}_1, \bar{z}_2 \in S_1 + S_2$

$\lambda \bar{z}_1 + (1-\lambda)\bar{z}_2 \in S_1 + S_2$

$\Rightarrow S_1 + S_2$ is convex set.

Ex The intersection of 2 convex sets is also convex.

→ let A and B be two convex sets show that $A \cap B$ is convex.

let $\bar{x}, \bar{y} \in A \cap B$

$\Rightarrow \bar{x}, \bar{y} \in A$ & $\bar{x}, \bar{y} \in B$

for $0 \le \lambda \le 1$ & A, B → convex set.

$\Rightarrow \lambda \bar{x} + (1-\lambda)\bar{y} \in A$ & $\lambda \bar{x} + (1-\lambda)\bar{y} \in B$

$\Rightarrow \lambda \bar{x} + (1-\lambda)\bar{y} \in A \cap B$

$\forall\ \bar{x}, \bar{y} \in A \cap B$

$\lambda \bar{x} + (1-\lambda)\bar{y} \in A \cap B$ for $0 \le \lambda \le 1$

$\therefore A \cap B$ is convex.

The intersection of two convex sets is also convex.

convexity

~~Conversity~~ them:

Ex the set of all convex combination of a finite number of points of $S \subset \mathbb{R}^n$ is a convex set.

sol^n: Let $C = \{ \bar{x} \mid \bar{x} = \sum_{i=1}^{m} \lambda_i \bar{x}_i ,\ \lambda_i \geq 0 ,\ \sum_{i=1}^{m} \lambda_i = 1 \}$,

Show that C is convex set.

Let $\bar{x}', \bar{x}'' \in C$.

with $\bar{x}' = \sum_{i=1}^{m} \lambda_i' \bar{x}_i$; $\bar{x}'' = \sum_{i=1}^{m} \lambda_i'' \bar{x}_i$

where, $\lambda_i', \lambda_i'' \geq 0$, $\sum_{i=1}^{m} \lambda_i' = 1 = \sum_{i=1}^{m} \lambda_i''$

Now, consider,

$\bar{\omega} = \lambda \bar{x}' + (1-\lambda)\bar{x}''$, for $0 \leq \lambda \leq 1$

$= \lambda \sum_{i=1}^{m} \lambda_i' \bar{x}_i + (1-\lambda) \sum_{i=1}^{m} \lambda_i'' \bar{x}_i$

$= \sum_{i=1}^{m} (\lambda \lambda_i' + (1-\lambda)\lambda_i'')\bar{x}_i$

$= \sum_{i=1}^{m} \mu_i \bar{x}_i$

where $\mu_i = \lambda \lambda_i' + (1-\lambda)\lambda_i''$

since for each $\lambda_i' \geq 0$, $\lambda_i'' \geq 0$ & $0 \leq \lambda \leq 1$

$\Rightarrow \lambda \lambda_i' + (1-\lambda)\lambda_i'' \geq 0$

$\Rightarrow \mu_i \geq 0$

and, $\sum_{i=1}^{m} \mu_i = \sum_{i=1}^{m} (\lambda \lambda_i' + (1-\lambda)\lambda_i'')$

$= \lambda \sum_{i=1}^{m} \lambda_i' + (1-\lambda) \sum_{i=1}^{m} \lambda_i''$

$= \lambda + 1 - \lambda$

$= 1$

$\omega = \sum_{i=1}^{m} \mu_i \bar{x}_i$

where $\mu_i \geq 0$, $\sum_{i=1}^{m} \mu_i = 1$

$\bar{\omega} \in S$.

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Thus, $\forall\ \bar{x}', \bar{x}'' \in C$

$\bar{\omega} := \lambda \bar{x}_1 + (1-\lambda)\bar{x}_{11} \in C$

C is Convex set. for $0 < \lambda \leq 1$

★ the general form of linear program problems.

★ find the value of desicion variables. $x_1, x_2 \ldots x_n$ that optimize (maximize or minimize)

$$z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

Subject to the Constraints

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \ (\leq, =, \geq) \quad \text{---}(1)$$

$$\left.\begin{array}{l} a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \ (\leq, =, \geq) \\ a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n \ (\leq, =, \geq) \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \ (\leq, =, \geq) \end{array}\right\} \text{---}(2)$$

and $x_i \geq 0$, $i = 1, 2 \ldots n$ ——(3)

in the above L.P problem each Constraint may take only one of the three since $\leq, =, \geq$.

→ the variables $x_1, x_2 \ldots x_n$ are called dision variables.

(1) is called objective f^n. the real numbers $c_1, c_2 \ldots c_n$ are called post Coefficients.

★ the standard form can be return in the matrix notation as below.

Optimize $z = cx_i$ ——(1)

Subject to $Ax\ (\leq, =, \geq)\ B$ ——(2)

and $x \geq 0$ ——(3)

where, $C = [c_1 \; c_2 \; c_3 \; \cdots \; c_n]_{1\times n}$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$

* Slack & surplus variables :- the constraint of the form $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \le b_i$

i.e. $\sum_{j=1}^{n} a_{ij}x_j \le b_i$; $i = 1, 2 \cdots m$

Can be converted to the equality

$$\sum_{j=1}^{n} a_{ij}x_j + s_i = b_i$$

$i = 1, 2 \cdots m.$

by adding variable $s_i \ge 0$; $i = 1, 2 \cdots n.$ the variable s_i are known as slack variables.

→ Surplus :- i.e. $\sum_{j=1}^{n} a_{ij}x_j \ge b_i$, $i = 1, 2 \cdots n.$

Can be converted to the equality

$$\sum_{j=1}^{n} a_{ij}x_j - s_i = b_i$$

$i = 1, 2 \cdots m.$