Unit : 1

Chapter 1

Limit-Continuity of function of several variables and partial derivatives

1.1 : Limit and continuity :

We have studied limit, continuity, differentiation of a function of one variable. Now we shall study them for function of more variables.

[1] Function of more than one variable :

Let $S \subset R^n$ $n \in N - \{1\}$, R is the set of all real numbers, then a function $f : S \to R$ such that $f(x) = u$, is called a real valued function of n variables, where

$x = \{x_1, x_2, x_3, ..., x_n\} \in S$ and $u \in R$.

It is also denoted by $f : x \to u$, $x \in S$.

e.g. Let $S = \{(x, y)/x^2 + y^2 \leq 1\}$, then a function $f : S \to R$ such that $f(x, y) = \sqrt{1 - x^2 - y^2}$ is a function of two variables whose domain is S and range is $[0, 1]$. Here we shall denote neighbourhood by nbhd.

[2] Spherical nbhd of $a \in R^n$:

Let $a = (a_1, a_2, a_3, ..., a_n) \in R^n$ be a fixed point and $x = (x_1, x_2, x_3, ..., x_n) \in R^n$ be a variable point and $\delta \in R^+$, then a set $\{x \in R^n \mid 0 \leq |x - a| < \delta\}$ is called a spherical nbhd. of a in R^n and is denoted by $N(a, \delta)$ or $N_\delta(a)$.

e.g. In R^2, $\{(x_1, x_2) \in R^2 \mid 0 \leq |(x_1, x_2) - (a_1, a_2)| < \delta\}$ is a spherical nbhd of $a = (a_1, a_2)$ which is a set of all point in R^2 which lie in the circle whose centre is at a and radius equal to δ.

Scanned by CamScanner

1.3 Illustrations :

Evaluate the following limits by definition if exist :

[1] $\lim_{(x,y)\to(2,3)} [3xy]$.

Solution : Here $(x, y) \to (2, 3) \Rightarrow (x, y) \in N'((2, 3), \delta)$

$$\Rightarrow \left|(x, y) - (2, 3)\right| < \delta$$

$$\Rightarrow |x-2| < \delta, \ |y-3| < \delta$$

$$\Rightarrow 2-\delta < x < 2+\delta, \ 3-\delta < y < 3+\delta$$

$$\Rightarrow 3(2-\delta)(3-\delta) < 3xy < 3(2+\delta)(3+\delta)$$

$$\Rightarrow 18 - 15\delta + 3\delta^2 < 3xy < 18 + 15\delta + 3\delta^2$$

$$\Rightarrow 18 - (15\delta + 3\delta^2) < 3xy < 18 + (15\delta + 3\delta^2)$$

$$\therefore \ 15\delta + 3\delta^2 = \varepsilon$$

$$\therefore \ 3\delta^2 + 15\delta - \varepsilon = 0$$

$$\Rightarrow \delta = \frac{-15 \pm \sqrt{225 + 12\varepsilon}}{6}$$

$$= -\frac{5}{2} + \sqrt{\frac{\varepsilon}{3} + \frac{25}{4}} \qquad [\because \delta > 0]$$

$\Rightarrow$ For every $N(18, \varepsilon)$, $\exists N((2, 3), \delta) \ni$
$(x, y) \in N'((2, 3), \delta) \cap D_f$
$\Rightarrow f(x, y) \in N(18, \varepsilon)$

$$\therefore \lim_{(x,y)\to(2,3)} f(x, y) = 18$$

i.e. $\lim_{(x,y)\to(2,3)} 3xy = 18$

Scanned by CamScanner

[2] $\lim_{(x,y)\to(2,1)} \frac{2x+y}{3y-x}$

Solution : Here $(x, y) \to (2, 1)$ and $f(x, y) = \frac{2x+y}{3y-x}$

$$\Rightarrow \ |(x, y)-(2,1)| < \delta$$

$$\Rightarrow \ 2-\delta < x < 2+\delta, \ 1-\delta < y < 1+\delta$$

$$\Rightarrow \ 4-2\delta < 2x < 4+2\delta, \ 3-3\delta < 3y < 3+3\delta$$

$$\Rightarrow \ 5-3\delta < 2x+y < 5+3\delta \ \text{ and}$$

$$1-2\delta < 3y-x < 1+2\delta$$

$$\Rightarrow \ \frac{1}{1+2\delta} < \frac{1}{3y-x} < \frac{1}{1-2\delta}$$

$$\Rightarrow \ \frac{5-3\delta}{1+2\delta} < \frac{2x+y}{3y-x} < \frac{5+3\delta}{1-2\delta}$$

$$\Rightarrow \ 5-\frac{13\delta}{1+2\delta} < \frac{2x+y}{3y-x} < 5+\frac{13\delta}{1-2\delta}$$

$$\Rightarrow \ 5-\frac{13\delta}{1-2\delta} < \frac{2x+y}{3y-x} < 5+\frac{13\delta}{1-2\delta}$$

$$\Rightarrow \ 5-\varepsilon < \frac{2x+y}{3y-x} < 5+\varepsilon$$

where $\varepsilon = \frac{13\delta}{1-2\delta} \Rightarrow \delta = \frac{\varepsilon}{2\varepsilon+13}, \ \forall \varepsilon > 0$

$$\Rightarrow \ |(x, y)-(2,1)| < \delta \Rightarrow |f(x, y)-5| < \varepsilon$$

$$\therefore \ \lim_{(x,y)\to(2,1)} \frac{2x+y}{3y-x} = 5$$

Scanned by CamScanner

[3] $f : R^2 \to R$ is the function such that

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2} \quad , \quad (x, y) \neq (0, 0)$$

$$= 3 \quad , \quad (x, y) = (0, 0) \text{ at point } (0, 0).$$

Solution : Let $(x, y) \to (0, 0) \Rightarrow f(x, y) \to l$

$$\therefore \quad (x, y) \in N'((0, 0), \delta) \Rightarrow \left|\frac{x^2 - y^2}{x^2 + y^2} - l\right| < \varepsilon$$

$$\Rightarrow \quad 0 < \sqrt{x^2 + y^2} < \delta \Rightarrow \left|\frac{x^2 - y^2}{x^2 + y^2} - l\right| < \varepsilon \qquad \text{...(1)}$$

Now take $\varepsilon = 1$, then for $0 < \sqrt{x^2 + y^2} < \delta$,

$$\left|\frac{x^2 - y^2}{x^2 + y^2} - l\right| < 1$$

Now for the point $\left(-\frac{\delta}{2}, 0\right)$, $0 < \sqrt{x^2 + y^2} < \delta$,

$$\left|\frac{(\delta/2)^2 - 0}{(\delta/2)^2 + 0} - l\right| < 1 \Rightarrow |1 - l| < 1 \qquad \text{...(2)}$$

and for the point $\left(0, \frac{\delta}{2}\right)$, $0 < \sqrt{x^2 + y^2} < \delta$,

$$\left|\frac{0 - (\delta/2)^2}{0 + (\delta/2)^2} - l\right| < 1 \Rightarrow |1 + l| < 1 \qquad \text{...(3)}$$

$\therefore$ Results (2) and (3),

$2 = |1 - l + 1 + l| \leq |1 - l| + |1 + l| < 1 + 1 = 2$

Which is absurd

$$\therefore \quad \lim_{(x, y) \to (0, 0)} \frac{x^2 - y^2}{x^2 + y^2} \text{ does not exist.}$$

Scanned by CamScanner

[4] $\lim_{(x,y)\to(1,1)} \frac{x^2+y^2}{x+y}$

Solution : Here let $(x, y) \to (1, 1) \Rightarrow (x, y) \in N'((1, 1), \delta)$

$$\Rightarrow \quad 0 < \sqrt{(x-1)^2 + (y-1)^2} < \delta$$

$$\Rightarrow \quad 0 < |x-1| < \delta, \ 0 < |y-1| < \delta$$

$$\Rightarrow \quad 1-\delta < x < 1+\delta, \quad 1-\delta < y < 1+\delta$$

$$\Rightarrow \quad 1-2\delta+\delta^2 < x^2 < 1+2\delta+\delta^2,$$

and $1-2\delta+\delta^2 < y^2 < 1+2\delta+\delta^2$

$$\Rightarrow \quad 2-4\delta+2\delta^2 < x^2+y^2 < 2+4\delta+2\delta^2$$

and $2-2\delta < x+y < 2+2\delta$ ...(1)

$$\Rightarrow \quad (2-4\delta+2\delta^2) - (2+2\delta) < (x^2+y^2) - (x+y) < 2+4\delta+2\delta^2 - (2-2\delta)$$

$$\Rightarrow \quad -6\delta + 2\delta^2 < x^2+y^2 - x - y < 6\delta + 2\delta^2$$

$$\Rightarrow \quad -(6\delta + 2\delta^2) < x^2+y^2 - x - y < 6\delta + 2\delta^2$$

$$\Rightarrow \quad -\frac{6\delta + 2\delta^2}{2+2\delta} < \frac{x^2+y^2-x-y}{x+y} < \frac{6\delta+2\delta^2}{2-2\delta}$$

$$\Rightarrow \quad -\frac{3\delta + \delta^2}{1+\delta} < \frac{x^2+y^2-x-y}{x+y} < \frac{3\delta+\delta^2}{1-\delta}$$

$$\Rightarrow \quad -\frac{3\delta+\delta^2}{1-\delta} < -\frac{3\delta+\delta^2}{1+\delta} < \frac{x^2+y^2-x-y}{x+y} < \frac{3\delta+\delta^2}{1-\delta}$$

$= \varepsilon$ say

$$\Rightarrow \quad -\varepsilon < \frac{x^2+y^2}{x+y} - 1 < \varepsilon$$

$$\Rightarrow \quad \left|\frac{x^2+y^2}{x+y} - 1\right| < \varepsilon, \text{ where } \varepsilon = \frac{3\delta+\delta^2}{1-\delta}.$$

Scanned by CamScanner

Exercise :

(1) $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x-y}$

(2) $\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$

(3) $\lim_{(x,y)\to(0,0)} \frac{x^5 y}{x^{10}+y^2}$

(4) $f(x,y)=\begin{cases} \frac{x^2+2y^2}{x^2-2y^2}; & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$

(5) $f(x,y)=\begin{cases} \frac{x-y}{x+y}; & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$

1.8 Continuity :

Let $f: E \subset R^n \to R$ be a function and $a \in E$ be fixed point, then the function f is said of be continuous at point $\boldsymbol{a}$, if $\lim_{\boldsymbol{x}\to \boldsymbol{a}} f(\boldsymbol{x}) = f(\boldsymbol{a})$. In particular $f: E \subset R^2 \to R$ *i.e.* $f(x, y)$ be a real function of two variables x and y defined on $E \subset R^2$ and if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$, then the function f is said to be continuous at point (a, b) where $\boldsymbol{x} = (x, y)$, $\boldsymbol{a} = (a, b)$.

NOTE : Here $\lim_{x\to a} f(x) = f(a)$ means.

(1) Function f is defined at point $\boldsymbol{a} \in E$ *i.e.* $f(\boldsymbol{a})$ exist,

(2) $\lim_{x\to a} f(x)$ exists.

(3) The values of $\lim_{x\to a} f(x)$ and $f(\boldsymbol{a})$ are equal.

Scanned by CamScanner

[2] $f(x, y) = \dfrac{x^2 - y^2}{x + y}$, $(x, y) \in (0, 0)$

$= 0$; $(x, y) = (0, 0)$, at $(0, 0)$.

Solution : Here $f(0, 0) = 0$

Define function $y = \phi(x) = 3x$ and $y = \psi(x) = x^2$

$$\therefore \quad \lim_{x \to 0} f(x, \phi(x)) = \lim_{x \to 0} \frac{x^2 + 9x^2}{x + 3x} = \lim_{x \to 0} \frac{5x}{2} = 0$$

$$\text{and} \quad \lim_{x \to 0} f(x, \psi(x)) = \lim_{x \to 0} \frac{x^2 + x^4}{x + x^2} = \lim_{x \to 0} \frac{x^2(1 + x^2)}{x(1 + x)} = 0$$

$$\therefore \quad \lim_{x \to 0} f(x, \phi(x)) = f(x, \psi(x)) = 0$$

$$\therefore \quad \lim_{(x, y) \to (0, 0)} f(x, y) = 0 = f(0, 0).$$

$\therefore$ The function f is continuous at point $(0, 0)$.

[3] $f(x, y) = \dfrac{xy^2}{(x^2 + y^2)}$, $(x, y) \neq (0, 0)$

$= 0$, $(x, y) = (0, 0)$ at point $(0, 0)$.

Solution : Here $f(x, y) = \dfrac{xy^2}{(x^2 + y^2)}$, $(x, y) \neq (0, 0)$

$$y = \phi(x) = m_i x, \; y = \psi(x) = x^2$$

$$\therefore \quad \lim_{x \to 0} f(x, \phi(x)) = \lim_{x \to 0} \frac{x \cdot m_i^2 x^2}{x^2 + m_i^2 x^2}$$

$$= \lim_{x \to 0} \frac{m_i^2 x}{1 + m_i^2} = 0$$

Scanned by CamScanner

and $\lim_{x\to 0} f(x, \psi(x)) = \lim_{x\to 0} \frac{x \cdot x^4}{x^2 + x^4}$

$$= \lim_{x\to 0} \frac{x^2}{1+x^2} = 0$$

$\therefore \quad \lim_{(x,y)\to(0,0)} f(x, y) = 0 = f(0,0)$

$\therefore$ The function f is continous at (0, 0).

[4] $f(x, y) = \dfrac{\sin x - \sin y}{\tan x - \tan y}$, $\tan x \neq \tan y$

$= 1$, $\tan x = \tan y$, at point (0, 0).

Solution : Here $f(0, 0) = 1$ ...(1)

Define function $y = \phi_i(x) = m_i x$

$\therefore \quad \lim_{x\to 0} f(x, \phi(x)) = \lim_{x\to 0} \frac{\sin x - \sin m_i x}{\tan x - \tan m_i x}$

$$= \lim_{x\to 0} \frac{2\sin(m_i - 1)\frac{x}{2} \cos\left(m_i + 1\right)\frac{x}{2}}{\sin(m_i - 1)\, x} \times \cos x \cdot \cos m_i x$$

$$= \lim_{x\to 0} \frac{\sin(m_i - 1)\frac{x}{2}}{(m_i - 1)\frac{x}{2}} \times \frac{(m_i - 1)x}{\sin\,(m_i - 1)\, x} \times 1 \times 1 \times 1$$

$= 1 \times 1 \times 1 = 1 =$ constant, for $i = 1, 2, 3, \ldots\ldots$

$\therefore \quad \lim_{(x,y)\to(0,0)} f(x, y) = 1$ ...(2)

$\therefore$ Results (1) and (2),

$\Rightarrow \quad \lim_{(x,y)\to(0,0)} f(x, y) = f(0,0) = 1$

$\Rightarrow$ The function f is continous at point (0, 0).

Scanned by CamScanner

Unit – II

Scanned by CamScanner

Unit : 2

Chapter 2

Differentiability of function of several variables – I

2.1 : Differentiation :

We have studied the derivative of a function of one variable. Now we shall study the derivative of a function of several variables with the given conditions.

Directional Derivative :

Let $f : E \subset R^n \to R$ be real valued function and if for x, $x + hu \in E$, $h \neq 0$, $\lim_{h\to 0} \frac{f(x+hu) - f(x)}{h}$ exists, then this limit is called the directional derivative of function $f(x)$ at point x along the direction of unit vector u and is denoted by $D_u f(x)$.

Thus, $D_u f(x) = \lim_{h\to 0} \frac{f(x+hu) - f(x)}{h}$.

Illustration :

Let $f(x, y) = \frac{xy^2}{x^2 + y^4}$, $x \neq 0, y \neq 0$

$= 0$, $x = 0, y = 0$, then

find directional derivative of function f at point (0, 0) along the direction of the vector $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

Scanned by CamScanner

Solution : Let $u = (u_1, u_2)$

$$\therefore \quad D_u f(0, 0) = \lim_{h\to 0} \frac{f((0,0) + h(u_1 \cdot u_2)) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{f(hu_1, hu_2) - 0}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{hu_1 \cdot h^2u_2^2}{h^2u_1^2 + h^4u_2^4}}{h}$$

$$= \lim_{h\to 0} \frac{u_1u_2^2}{u_1^2 + h^2u_2^4} = \frac{u_2^2}{u_1}$$

If $u_1 = 0$, then $D_u f(0, 0) = \lim_{h\to 0} \frac{f(0, hu_2) - f(0,0)}{h}$

$$= \lim_{h\to 0} \frac{0 - 0}{h} = 0$$

Here $u = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$\therefore$ Required directional derivative is $\frac{(1/\sqrt{2})^2}{(1/\sqrt{2})} = \frac{1}{\sqrt{2}}$.

NOTE : The directional derivative of function f exists at the given point in its domain even yet the function f may or may not be continuous at that point.

Scanned by CamScanner

[3] **Evaluate f_x and f_y for the following functions and determine the valules of $x f_x + y f_y$.**

(1) $$f(x, y) = \frac{x^2 + y^2}{x + y}, \quad x + y \neq 0$$
$$= 0, \quad x + y = 0$$

Solution : $$f(x, y) = \frac{x^2 + y^2}{x + y}$$

$$\Rightarrow f_x(x, y) = \frac{(x + y)2x + (x^2 + y^2)1}{(x + y)^2}$$

$$= \frac{x^2 + 2xy - y^2}{(x + y)^2}$$

and $$f_y(x, y) = \frac{(x + y)\, 2y + (x^2 + y^2)\, 1}{(x + y)^2}$$

$$= \frac{y^2 + 2xy - x^2}{(x + y)^2}$$

$$\therefore \; x f_x + y f_y = \frac{x(x^2 + 2xy - y^2) + y(y^2 + 2xy - x^2)}{(x + y)^2}$$

$$= \frac{(x + y)(x^2 + y^2)}{(x + y)^2}$$

$$= \frac{x^2 + y^2}{x + y}$$

Scanned by CamScanner

(2) $f(x, y) = \sin^{-1} \dfrac{x}{y}, \quad y \neq 0$

$\quad = 1 \qquad , \quad y = 0.$

Solution : Here $f(x, y)\ \sin^{-1}\dfrac{x}{y}$, $0 < |x| < |y|$

$$\Rightarrow \qquad f_x = \frac{1}{\sqrt{1 - \dfrac{x^2}{y^2}}} \times \frac{1}{y} = \frac{x}{\sqrt{y^2 - x^2}}$$

and

$$f_y = \frac{1}{\sqrt{1 - \dfrac{x^2}{y^2}}} \left[-\frac{x}{y^2}\right] = \frac{x}{y\sqrt{y^2 - x^2}}$$

$$\therefore \quad x f_x + y f_y = \frac{x}{\sqrt{y^2 - x^2}} - \frac{y \cdot x}{y\sqrt{y^2 - x^2}} = 0$$

(3) $f(x, y) = \sin^{-1} \dfrac{x^2 + y^2}{x + y}, \quad x + y \neq 0$

$\quad = \dfrac{\pi}{4} \qquad , \quad x + y = 0.$

Solution : $f(x, y) = \sin^{-1} \dfrac{x^2 + y^2}{x + y}$

$$\Rightarrow \qquad f_x(x, y) = \frac{1}{\sqrt{1 - \left(\dfrac{x^2 + y^2}{x + y}\right)^2}} \times \frac{x^2 + 2xy - y^2}{(x + y)^2}$$

(From Ex. 2)

Scanned by CamScanner

$$= \frac{x^2 + 2xy - y^2}{\sqrt{(x+y)^2 - (x^2+y^2)^2}\,(x+y)}$$

and $\quad f_y(x, y) = \dfrac{1}{\sqrt{1 - \dfrac{(x^2+y^2)^2}{(x+y)^2}}} \times \dfrac{y^2 + 2xy - x^2}{(x+y)^2}$

$$= \frac{y^2 + 2xy - x^2}{(x+y)\sqrt{(x+y)^2 - (x^2+y^2)^2}}$$

$\therefore \quad x f_x(x, y) + y f_y(x, y)$

$$= \frac{x(x^2 + 2xy - y^2) + y(y^2 + 2xy - x^2)}{(x+y)\sqrt{(x+y)^2 - (x^2+y^2)^2}}$$

$$= \frac{(x+y)(x^2+y^2)}{(x+y)\sqrt{(x+y)^2 - (x^2+y^2)^2}}$$

$$= \frac{x^2+y^2}{\sqrt{(x+y)^2 - (x^2+y^2)^2}}$$

Scanned by CamScanner

2.4 Partial Derivative (Differential Coefficient) of higher order :

If $E \subset R^n$ is a non-empty open subset and $f : E \to R$ is a real valued function whose partial derivatives $D_i f : E \to R$ exist, $i = 1, 2, 3, 4, \ldots, n$. If partial derivatives of $D_i f$ exist, they are called partial derivatives of second order and so on. In particular $f : E \subset R^3 \to R$ is a real valued function of three variables x, y, z the partial derivatives of first order are.....

$$D_1 f = \frac{\partial f}{\partial x},\ D_2 f = \frac{\partial f}{\partial y},\ D_3 f = \frac{\partial f}{\partial z}$$

The partial derivatives of second order are.....

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} = f_{11} = D_{11} f$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy} = f_{22} = D_{22} f$$

$$\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial^2 f}{\partial z^2} = f_{zz} = f_{33} = D_{33} f$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy} = f_{12} = D_{12} f$$

Scanned by CamScanner

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{yx} = f_{21} = D_{21}f$$

$$\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial z \partial x} = f_{xz} = f_{13} = D_{13}f$$

$$\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial z \partial y} = f_{yz} = f_{23} = D_{23}f \quad \text{etc.}$$

Now, $f_{xy}(x, y) = (f_x(x, y)) = \lim_{k \to 0} \frac{f_x(x, y+k) - f_x(x, y)}{k}$

But $f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$

$$\lim_{h \to 0} \frac{f(x+h, y+k) - f(x, y+k)}{h}$$

$$\therefore \quad f_x(x, y) = \lim_{k \to 0} \frac{\lim_{h \to 0} \frac{f(x+h, y+k) - f(x, y+k)}{h} - \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}}{k}$$

$$= \lim_{k \to 0} \lim_{h \to 0} \frac{f(x+h, y+k) - f(x, y+k) - f(x+h, y) + f(x, y)}{hk} \quad \text{...(1)}$$

Similarly,

$$f_{yx}(x, y) = \lim_{h \to 0} \lim_{k \to 0} \frac{f(x+h, y+k) - f(x, y+k) - f(x+h, y) + f(x, y)}{hk} \quad \text{...(2)}$$

$$f_{xx} = \lim_{h \to 0} \lim_{h \to 0} \frac{f(x+2h, y) - 2f(x+h, y) + f(x, y)}{h^2} \quad \text{...(3)}$$

Scanned by CamScanner

and

$$f_{yy} = \lim_{k\to 0} \lim_{k\to 0} \frac{f(x, y+2k) - 2f(x, y+k) + f(x, y)}{k^2} \qquad \text{...(4}$$

From (1),

$$f_{xy}(0,\ 0) = \lim_{k\to 0} \lim_{h\to 0} \frac{f(h, k) - f(0, k) - f(h, 0) + f(0, 0)}{hk}$$

From (2),

$$f_{yx}(0,\ 0) = \lim_{h\to 0} \lim_{k\to 0} \frac{f(h, k) - f(0, k) - f(h, 0) + f(0, 0)}{hk}$$

From (3),

$$f_{xx}(0,\ 0) = \lim_{h\to 0} \lim_{h\to 0} \frac{f(2h, 0) - 2f(h, 0) + f(0, 0)}{h^2}$$

OR

$$= \lim_{h\to 0} \frac{f_x(h, 0) - f_x(0, 0)}{h}$$

From (4),

$$f_{yy}(0,\ 0) = \lim_{k\to 0} \lim_{k\to 0} \frac{f(0, 2k) - 2f(0, k) + f(0, 0)}{k^2}$$

$$= \lim_{k\to 0} \frac{f_y(0, k) - f_y(0, 0)}{k}$$

Scanned by CamScanner

[2] **Find $f_{xy}(x, y)$, $f_{yx}(x, y)$ where $(x, y) \neq (0, 0)$, for the following function :**

(1) $$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}, \quad (x, y) \neq (0, 0)$$

$$= 0 \quad , (x, y) = (0, 0).$$

Solution : Here

$$f_x(x, y) = xy \cdot \frac{(x^2 + y^2)(2x) - (x^2 - y^2)(2x)}{(x^2 + y^2)^2} + y \cdot \frac{x^2 - y^2}{x^2 + y^2}$$

$$= \frac{4x^2y^3 + x^4y - y^5}{(x^2 + y^2)^2}$$

$$\Rightarrow f_{xy}(x, y) = \frac{(x^2 + y^2)^2(12x^2y^2 + x^4 - 5y^4) - (4x^2y^3 + x^4y - y^5)2 \cdot 2x(x^2 + y^2)}{(x^2 + y^2)^4}$$

$$= \frac{(x^2 + y^2)(12x^2y^2 + x^4 - 5y^4) - 4x(4x^2y^3 + x^4y - y^5)}{(x^2 + y^3)^4}$$

$$= \frac{9x^4y^2 - 9x^2y^4 + x^6 - y^6}{(x^2 + y^2)^3}$$

Similarly, $f_y(x, y) = \dfrac{x^5 - xy^4 - 4x^3y^2}{(x^2 + y^2)^2}$ and

$$f_{yx}(x, y) = \frac{9x^4y^2 - 9x^2y^4 + x^6 - y^6}{(x^2 + y^2)^3}$$

Thus $\quad f_{xy}(x, y) = f_{yx}(x, y)$

Scanned by CamScanner

Theorem 1 :

If function $z = f(x, y)$, defined on an open set $E \subset R^2$, is differentiable at point $(x, y) \in E$, then its partial derivatives f_x and f_y exist at point (x, y).

Proof : Since $z = f(x, y) \Rightarrow z + \delta z = f(x + \delta x, y + \delta y)$

Also function f is differentiable at point (x, y).

$\therefore \quad f(x + \delta x, y + \delta y) = f(x, y) + A\delta x + B\delta y + \varepsilon \varrho$

where A and B are independent of $\delta x, \delta y$

and $\varepsilon \to 0$ as $\varrho = \sqrt{\delta x^2 + \delta y^2} \to 0$ ...(I)

Case (i) : If we put $\delta y = 0, \delta x \neq 0$ in (1) then

$$f(x + \delta x, y) - f(x, y) = A\delta x + \varepsilon \delta x$$

$$\Rightarrow \quad \frac{f(x+\delta x, y) - f(x, y)}{\delta x} = A + \varepsilon, \ \delta x \neq 0$$

$$\Rightarrow \quad f_x = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}$$

$$= \lim_{\delta x \to 0 \ \delta y \to 0} [A + \varepsilon] = \lim_{\rho \to 0} [A + \varepsilon] = A + 0 = A$$

Case (ii) : If we put $\delta x = 0, \delta y \neq 0$ in (1), then

$$f(x, y + \delta y) - f(x, y) = B\delta y + \varepsilon \delta y$$

$$\Rightarrow \quad \frac{f(x, y+\delta y) - f(x, y)}{\delta y} = B + \varepsilon$$

$$\Rightarrow \quad \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x, y)}{\delta y} = \lim_{\delta y \to 0 \ \delta x \to 0} [B + \varepsilon]$$

$$\Rightarrow \quad f_y = \lim_{\rho \to 0} [B + \varepsilon] = B + 0 = B$$

$\therefore \quad f_x, f_y$ exist at point (x, y).

Scanned by CamScanner

Theorem 2 :

If a function $z = f(x, y)$, defined on an open set $E \subset R^2$, is differentiable at point $(a, b) \in E$, then the function f is continuous at $(a, b) \in E$.

Proof : Since $z = f(x, y)$ is differentiable function at point $(a, b) \in E \subset R$.

$\therefore \quad f(a + h, b + k) = f(a, b) + f_x(a, b)h + f_y(a, b)k + \varepsilon \rho$

where $\varepsilon \to 0$ as $\rho \to 0$ ...(1)

$$\therefore \lim_{(x, y) \to (a, b)} f(x, y) = \lim_{h \to 0 \; k \to 0} f(a + h, b + k)$$

$$= \lim_{h \to 0 \; k \to 0} [f(a, b) + f_x(a, b)h + f_y(a, b)k + \varepsilon\rho]$$

$$= f(a, b) + f_x(a, b) \cdot 0 + f_y(a, b) \cdot 0 + 0$$

$$= f(a, b)$$

$\therefore$ The function f is continuous at point (a, b).

NoTE : The converse of this theorem is not true *i.e.* If the function f is continous at $(a, b) \in D_f$, it may or may not be differentiable at point (a, b).

Illustration : Consider the function

$$f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}}, \quad (x, y) \neq (0, 0)$$

$$= 0 \qquad , (x, y) = (0, 0) \text{ at point } (0, 0).$$

Solution : Reader can easily varify that

$$f_x(0,0) = 0, \; f_y(0, 0) = 0$$

$$\lim_{x \to 0} f(x, \phi(x)) = \lim_{x \to 0} f(x, \psi(x)) = 0$$

$$\Rightarrow \lim_{(x, y) \to (0, 0)} = 0 = f(0, 0)$$

$\therefore$ This function f is continuous at point $(0, 0)$.

Scanned by CamScanner

2.7 Illustrations :

[1] **Consider the function**

$$f(x, y) = \frac{x^2 y^2}{x^2 + y^2}, \quad (x, y) \neq (0, 0)$$

$$= 0 \quad , (x, y) = (0, 0) \text{ at point } (0, 0).$$

Solution : Reader can easily verify that $f_x(0, 0) = 0$, $f_y(0, 0) = 0$ and the function f is continous at point (0, 0).

$$\therefore \quad \lim_{\varrho \to 0} \varepsilon = \lim_{\varrho \to 0} \frac{f(0+h, 0+k) - f(0,0) - f_x(0,0)h - f_y(0,0)k}{\varrho}$$

$$= \lim_{\varrho \to 0} \frac{f(h, k) - 0 - 0 \cdot h - 0 \cdot k}{\varrho}$$

$$= \lim_{h \to 0} \frac{h^2 k^2}{(h^2 + k^2)^{3/2}} \qquad (\because \varrho = \sqrt{h^2 + k^2})$$

Define $y = \phi_i(x) = m_i x \Rightarrow k = m_i h$

$$= \lim_{h \to 0} \frac{h^4 \cdot m_i^2}{h^3 (1 + m_i)^{3/2}} = \lim_{h \to 0} \frac{h m_i^2}{(1 + m_i)^{3/2}} = 0$$

$\therefore$ The function f is differentiable of point (0, 0).

[2] **Consider the function**

$$f(x, y) = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right), \quad (x, y) \neq (0, 0)$$

$$= 0 \quad , (x, y) = (0, 0)$$

at point (0, 0).

Solution : $f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h}$

$$= \lim_{h \to 0} \frac{h^2 \tan^{-1}(0) - 0 \tan^{-1}\left(\frac{h}{0}\right) - 0}{h}$$

$$= \lim_{h \to 0} \frac{0-0}{h} = 0$$

and $\quad f_y(0, 0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k}$

$$= \lim_{k \to 0} \frac{0 \tan^{-1} \frac{k}{0} - k^2 \tan^{-1}(0) - 0}{k}$$

$$= \lim_{k \to 0} \frac{0}{k} = 0$$

$\therefore \quad f_x(0, 0) = f_y(0, 0) = 0$

Now $f(h, k) = h f_x(0, 0) + k f_y(0, 0) + \varepsilon \varrho$

$\Rightarrow \quad h^2 \tan^{-1} \frac{k}{h} - k^2 \tan^{-1} \frac{h}{k} = h \cdot 0 + k \cdot 0 + \varepsilon \varrho$

$\therefore \quad \varepsilon = \frac{1}{\varrho}\left(h^2 \tan^{-1} \frac{k}{h} - k^2 \tan^{-1} \frac{h}{k}\right)$

Put $h = \varrho \cos \theta$, $k = \varrho \sin \theta$

$$= \frac{\varrho^2}{\varrho}[\cos^2\theta \cdot \tan^{-1} \tan\theta - \sin^2\theta \tan^{-1} \cot\theta]$$

$$= \varrho\left[\theta \cos^2\theta - \left(\frac{\pi}{2} - \theta\right)\sin^2\theta\right]$$

$\therefore \quad \varrho \to 0 \Rightarrow \varepsilon \to 0$

$\therefore \quad$ The function f is differential at point (0, 0).

Scanned by CamScanner

Discuss the differentiabiliy of the following function :

[3] $$f(x, y) = \frac{xy^4}{(x^2 + y^2)^2}, \quad (x, y) \neq (0, 0)$$

$$= 0 \qquad , (x, y) = (0, 0)$$

Solution : $f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h}$

$$= \lim_{h \to 0} \frac{0 - 0}{h} = 0$$

$$f_y(0, 0) = \lim_{k \to 0} \frac{f(0, k) - f(0, 0)}{k}$$

$$= \lim_{k \to 0} \frac{0 - 0}{k} = 0$$

Now, $f(h, k) - f(0, 0) = hf_x(0, 0) + kf_y(0, 0) + \varepsilon\varrho$

$\therefore \quad \frac{hk^4}{(h^2 + k^2)^2} - 0 = h \cdot 0 + k \cdot 0 + \varepsilon\varrho = \varepsilon\sqrt{h^2 + k^2}$

$\Rightarrow \quad \varepsilon = \frac{hk^4}{(h^2 + k^2)^{5/2}}$

Put $h = \varrho\cos\theta$, $k = \varrho\sin\theta$

$\Rightarrow \quad \varepsilon = \frac{\varrho^5 \cos\theta \sin^4\theta}{\varrho^5}$

$$= \cos\theta \sin^4\theta$$

which is independent of ϱ. $\therefore$ As $\varrho \to 0 \Rightarrow \varepsilon \nrightarrow 0$.

$\therefore$ $f(x, y)$ is not differentiable at point (0, 0).

[4] $f(x, y) = \dfrac{x^3 y^3}{(x^2 + y^2)^3}$, $(x, y) \neq (0, 0)$

$= 0$, $(x, y) = (0, 0)$.

Solution : $f_x(0, 0) = \lim_{h \to 0} \dfrac{f(h,0) - f(0,0)}{h}$

$$= \lim_{h \to 0} \frac{0-0}{h} = 0$$

and $f_y(0, 0) = \lim_{k \to 0} \dfrac{f(0,k) - f(0,0)}{k}$

$$= \lim_{k \to 0} \frac{0-0}{k} = 0$$

Now $f(h, k) - f(0, 0) = hf_x(0, 0) + kf_y(0, 0) + \varepsilon \varrho$

$$\therefore \quad \frac{h^3 k^3}{(h^2 + k^2)^3} - 0 = h \cdot 0 + k \cdot 0 + \varepsilon \varrho$$

$$= \varepsilon \sqrt{h^2 + k^2}$$

$$\Rightarrow \quad \varepsilon = \frac{h^3 k^3}{(h^2 + k^2)^{7/2}}$$

Put $h = \varrho \cos \theta$, $k = \varrho \sin \theta$

$$\Rightarrow \quad \varepsilon = \frac{\varrho^6 \cos^3 \theta \sin^4 \theta}{\varrho^7} = \frac{\cos^3 \theta \sin^3 \theta}{\varrho}$$

As $\varrho \to 0 \Rightarrow \varepsilon \not\to 0$.

$\therefore$ The function is not differentiable at point (0, 0).

Scanned by CamScanner

2.9 Equality of f_{xy} and f_{yx} :

Theorem 7 : Young's theorem :

Let f be a real function defined on non-empty open set $E \subset R^2$. If f_x, f_y exist in some nbhd. of (x, y) and both are differentiable at point (x, y) with respect to x and y then f_{xy} = f_{yx} at point (x, y).

Proof : Let $\Delta^2 f = f(x + h, y + k) - f(x + h, y) - f(x, y + k) + f(x, y)$

and let $\phi(x, y) = f(x, y + k) - f(x, y)$

and $\psi(x, y) = f(x + h, y) - f(x, y)$

then $\Delta^2 f = \phi(x + h, y) - \phi(x, y)$ ...(1)

and $\Delta^2 f = \psi(x, y + k) - \psi(x, y)$ ...(2)

Since f_x exists in some nbhd of $(x; y)$, so ϕ_x exists in some nbhd. of (x, y).

∴ By Lagrange's Mean Value theorem

$\phi(x + h, y) - \phi(x, y) = h\phi_x(x + \theta_1 h, y)$, $\theta_1 \in (0, 1)$ ...(3)

∴ Results (1) and (3) ⇒

$\Delta^2 f = h\phi_x(x + \theta_1 h, y)$, $\theta_1 \in (0, 1)$

$= h[f_x(x + \theta_1 h, y + k) - f_x(x + \theta_1 h, y)$ ...(4)

But f_x is differentiable at point (x, y).

∴ By theorem 3

$f_x(x + \theta_1 h, y + k) - f_x(x, y) = \theta_1 h f_{xx} + k f_{xy} + \varepsilon_1 \rho_1$

where $\varepsilon_1 \Rightarrow 0$ as $\rho_1 = \sqrt{(\theta_1 h)^2 + k^2} \to 0$ i.e. $h \to 0, k \to 0$

...(5)

and $f_x(x + \theta_1 h, y) - f(x, y) = \theta_1 h f_{xx} + 0 \cdot f_{xy} + \varepsilon_2 \rho_2$.

where $\varepsilon_2 \to 0$ as $\rho_2 = |\theta_1 h| \to 0$ i.e. $h \to 0$ ...(6)

Scanned by CamScanner

$\therefore$ Results (5) and (6) $\Rightarrow$

$$f_x(x + \theta_1 h, y + k) - f_x(x + \theta_1 h, y) = k f_{xy} + \varepsilon_1 \varrho_1 - \varepsilon_2 \varrho_2$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $h \to 0, k \to 0$

$$= k f_{xy} + \varepsilon' \varrho \text{ say where } \varepsilon' \to 0$$

as $\varrho = \sqrt{h^2 + k^2} \to 0$ ...(7)

$\therefore$ Results (4) and (7) $\Rightarrow$

$$\Delta^2 f = h[k + \varepsilon' \varrho]$$

$$= hk f_{xy} + h\varepsilon' \varrho$$

$$\therefore \quad \frac{\Delta^2 f}{hk} = f_{xy} + \frac{\varepsilon \varrho}{k}$$

$\therefore$ $\varepsilon'' \to 0$ as $\varrho \to 0$ *i.e.* $h \to 0, k \to 0$ ...(8)

Similarly, $\Delta^2 f = \psi(x, y + k) - \psi(x, y)$

$$= hk f_{yx} + k\varepsilon'' \varrho$$

where $\varepsilon'' \to 0$ as $\varrho = \sqrt{h^2 + k^2} \to 0$

$$\therefore \quad \lim_{h \to 0\, k \to 0} \frac{\Delta^2 f}{hk} = \lim_{h \to 0\, k \to 0} \left(f_{yx} + \frac{\varepsilon'' \varrho}{h} \right)$$

$$= f_{yx} \qquad (\because \varepsilon'' \to 0 \text{ as } \varrho \to 0) \qquad ...(9)$$

$\therefore$ Results (8) and (9) $\Rightarrow f_{xy} = f_{yx}$

Theorem 2 : Schartz's theorem :

Let $f : E \subset R^2 \to R$ be a function such that is its partial derivatives f_x, f_y, f_{xy} exist and are continuous in a nbhd. of a point (x, y), then f_{yx} exists such that $f_{xy} = f_{yx}$.

Proof : Let $\Delta^2 f = f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x, y)$

and $\phi(x, y) = f(x, y+k) - f(x, y)$

then $\Delta^2 f = \phi(x+h, y) - \phi(x, y)$ ...(1)

Since f_x exists in a nbhd. of point (x, y) say N,

$\therefore$ ϕ_x exists in the nbhd. N of point (x, y).

$\therefore$ By L. M. V. Theorem

$\phi(x+h, y) - \phi(x, y) = h\phi_x(x+\theta h, y), \theta \in (0, 1)$ ...(2)

$\therefore$ Results (1) and (2) $\Rightarrow$

$$\Delta^2 f = \phi_x(x+\theta h, y), \theta \in (0, 1)$$
$$= h(f_x(x+\theta h, y+k) - f_x(x+\theta h, y)), \theta \in (0, 1)$$

Now f_{xy} exists in N

$\therefore$ By L. M. V. theorem

$\Delta^2 f = h[kf_{xy}(x+\theta h, y+\theta_1 k), \theta, \theta_1 \in (0, 1)$ ...(3)

Now f_{xy} is continuous in N.

$$\lim_{h\to 0\, k\to 0} f_{xy}(x+\theta h, y+\theta_1 k) = f_{xy}$$

$\therefore$ $f_{xy}(x+\theta h, y+\theta_1 k) = f_{xy} + \varepsilon$

where $\varepsilon \to 0$ as $h \to 0$, $k \to 0$ ...(4)

$\therefore$ Results (3) and (4) $\Rightarrow$

$\Delta^2 f = hk[f_{xy} + \varepsilon]$ ...(5)

$\therefore$ Results (1) and (5) $\Rightarrow$

$$[f(x+h, y+k) - f(x+h, y)] - [f(x, y+k) - f(x, y)]$$
$$= hk[f_{xy} + \varepsilon]$$

Scanned by CamScanner

$$\therefore \quad \lim_{k \to 0}\left[\frac{f(x+h, y+k) - f(x+h, y)}{k} - \frac{f(x, y+k) - f(x, y)}{k}\right]$$

$$= \lim_{k \to 0} h[f_{xy} + \varepsilon]$$

$$\therefore \quad f_y(x + h, y) - f_y(x, y) = h f_{xy} + \varepsilon' h$$

where $\varepsilon' \to 0$ as $h, k \to 0$

$$\therefore \quad \lim_{h \to 0} \frac{f_y(x+h, y) - f_y(x, y)}{h} = \lim_{h \to 0} (f_{xy} + \varepsilon')$$

where $\varepsilon' \to 0$ as $h, k \to 0$

Since f_{xy} exists, so limit of L. H. S. exists, in that case it is nothing but limit $(f_y)_x$ of L.H.S. *i.e.* f_{yx} exists.

$$\therefore \quad f_{xy} = f_{yx}$$

NOTE : The converse of Shwartz's theorem is not true.

Consider the function

$$f(x, y) = \frac{x^2 y^2}{x^2 + y^2}, \quad (x, y) = (0, 0)$$

$$= 0, \quad (x, y) = (0, 0) \text{ at point } (0, 0).$$

We have seen by example of Note previously.

$$f_x(0, 0) = 0, \; f_y(0, 0) = 0$$

and $f_{xy}(0, 0) = 0, \; f_{yx}(0, 0) = 0$

$$\Rightarrow \; f_{xy}(0, 0) = f_{yx}(0, 0)$$

Also, $f_{xx}(0, 0) = \lim_{h \to 0} \dfrac{f_x(h, 0) - f_x(0, 0)}{h} = \lim_{h \to 0} \dfrac{0 - 0}{h} = 0$

Now, $f_{xy}(x, y) = \dfrac{\partial}{\partial y}\left[\dfrac{2xy^4}{(x^2 + y^2)^2}\right]$

$$= \frac{(x^2 + y^2)\, 8xy^3 - 2xy^4 \cdot 4y(x^2 + y^2)}{(x^2 + y^2)^4}$$

Scanned by CamScanner

$$= \frac{8xy^3(x^2+y^2) - 8xy^5}{(x^2+y^2)^3}$$

$$= \frac{8x^3y^3}{(x^2+y^2)^3}$$

Similarly, $f_{yx}(x, y) = \dfrac{8x^3y^3}{(x^2+y^2)^3}$

and $\lim_{x\to 0} f_{xy} = (x, m_i x) = \lim_{x\to 0} \dfrac{8m_i^3 x^6}{x^6(1+m_i^2)^2}$

$$= \lim_{x\to 0} \frac{8m_i^3}{(1+m_i^2)^3}$$

$$= \frac{8m_i^3}{1+m_i^2}$$ which is not finite.

$\therefore \quad f_{xy}$ is not continuous at (0, 0).

NOTE : $f_{xy}(h, k) = h f_{xyx}(0, 0) + k f_{xyy}(0, 0) + \varepsilon \varrho$

But $\quad f_{xyx}(0, 0) = \lim_{h\to 0} \dfrac{f_{xy}(h, 0) - f_{xy}(0, 0)}{h}$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

Similarly, $f_{xyy}(0, 0) = 0$

$f_{xy}(h, k) = \varepsilon \varrho$

$$\Rightarrow \quad \frac{8h^3k^3}{(h^2+k^2)^3} = \varepsilon \varrho$$

$$\Rightarrow \quad \varepsilon = \frac{1}{\varrho} \cdot \frac{8h^3k^3}{(h^2+k^2)^3}$$

Scanned by CamScanner

[1] If $z = f(ax + by) + g(ax - by)$, then prove that

$$b^2 \frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}$$

Solution : Since $z = f(ax + by) + g(ax - by) = f(u) + g(v)$

where $u = ax + by, \quad v = ax - by$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{du}{dx} + \frac{\partial g}{\partial v} \cdot \frac{dv}{dx} = \frac{\partial f}{\partial u} \cdot a + \frac{\partial g}{\partial v} \cdot a$$

$$\Rightarrow \quad \frac{\delta^2 z}{\delta y^2} = \frac{\partial^2 f}{\partial u^2} a^2 + \frac{\partial^2 g}{\partial v^2} a^2 = a^2 \left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} \right) \quad ...(1)$$

Similarly, $$\frac{\partial^2 z}{\partial y^2} = b^2 \left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} \right) \quad ...(2)$$

$\therefore$ Results (1) and (2) $\Rightarrow b^2 \frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}$

[2] If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, then show that

$$\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right]^2 u = 9(x^2 + y^2 + z^2)^{-2}$$

Solution : Since $u = \log(x^3 + y^3 + z^3 - 3xyz)$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz}$$

Scanned by CamScanner

$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3zx}{x^3 + y^3 + z^3 - 3xyz} \quad \text{and}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 2xy}{x^3 + y^3 + z^3 - 3xyz}$$

$$\Rightarrow \quad \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$= \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{3}{(x+y+z)}$$

$$\Rightarrow \quad \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u = \frac{\partial}{\partial x}\left(\frac{3}{(x+y+z)}\right) = -\frac{3}{(x+y+z)^2}$$

Similarly, $\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u$

$$= -\frac{3}{(x+y+z)^2}$$

$$\Rightarrow \quad \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u$$

$$+ \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u + \frac{\partial}{\partial z}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u$$

$$= \frac{-9}{(x+y+z)^2}$$

Scanned by CamScanner

[3] If $H = f(y - z, z - x, x - y)$, then prove that

$$\frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} + \frac{\partial H}{\partial z} = 0 \text{ .....}$$

Solution : Let $H = f(u, v, w)$, $u = y - z$; $v = z - x$, $w = x - y$

$$\therefore \quad \frac{\partial H}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial x}$$

$$= \frac{\partial f}{\partial u} \cdot 0 + \frac{\partial f}{\partial v} \cdot (-1) + \frac{\partial f}{\partial w} \cdot (1)$$

$$= -\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w} \quad \text{...(1)}$$

Similarly, $\frac{\partial H}{\partial y} = -\frac{\partial f}{\partial w} + \frac{\partial f}{\partial u}$ and $\frac{\partial H}{\partial z} = -\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}$ ...(2)

$$\therefore \quad \frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} + \frac{\partial H}{\partial z} = 0 \text{ [by results (1) and (2)]}$$

[4] **If $z = f(x, y)$ and $x = e^{-u} + e^{v}$, $y = e^{u} + e^{-v}$, then prove that**

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = y\frac{\partial z}{\partial y} - x\frac{\partial z}{\partial x}.$$

Solution : Here, $x = e^{-u} + e^{v}$, $y = e^{u} + e^{-v}$

$$\Rightarrow \quad \frac{\partial x}{\partial u} = -e^{-u}, \quad \frac{\partial x}{\partial v} = e^{v}, \quad \frac{\partial y}{\partial u} = e^{u}, \quad \frac{\partial y}{\partial v} = -e^{-v}$$

$$\therefore \quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}(-e^{-u}) + \frac{\partial z}{\partial y} e^{u} \quad \text{...(1)}$$

Similarly, $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} e^{v} + \frac{\partial z}{\partial y}(-e^{-v})$ ...(2)

Scanned by CamScanner

Results (1) and (2)

$$\Rightarrow \frac{\partial z}{\partial u}-\frac{\partial z}{\partial v} = -e^{-u}\frac{\partial z}{\partial x}+e^{u}\frac{\partial z}{\partial y}-e^{v}\frac{\partial z}{\partial x}+e^{-v}\frac{\partial z}{\partial y}$$

$$= \frac{\partial z}{\partial y}(e^{u}+e^{-v}) - \frac{\partial z}{\partial x}(e^{-u}+e^{v})$$

$$= y\frac{\partial z}{\partial y} - x\frac{\partial z}{\partial x}$$

[5] **If $u = \log(x^2+y^2)$, then prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$**

i.e. u harmonic function of x and y.

Solution : Let $u = f(r) = \log r^2 = 2\log r$, where $r = \sqrt{x^2+y^2}$

$$\Rightarrow \quad f'(r) = \frac{2}{r}, \quad f''(r) = -\frac{2}{r^2}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = f''(r)+\frac{1}{r}f'(r)$$

$$= -\frac{2}{r^2}+\frac{1}{r}\left(\frac{2}{r}\right) = 0$$ Hence the result.

[6] **If $x = r\cos\theta$, $y = r\sin\theta$ and $u = f(x, y)$, then prove that**

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial u}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2 u}{\partial \theta^2}.$$

Solution : Here $x = r\cos\theta$, $y = r\sin\theta$

$$\Rightarrow \quad r = \sqrt{x^2+y^2}, \ \theta = \tan^{-1}\frac{y}{x}$$

Scanned by CamScanner

$$\Rightarrow \quad \frac{\partial x}{\partial r} = \cos\theta, \frac{\partial x}{\partial \theta} = -r\sin\theta, \frac{\partial y}{\partial r} = \sin\theta, \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta, \ \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$$

$$\therefore \quad \frac{\partial \theta}{\partial x} = -\frac{y}{x^2+y^2} = \frac{r\sin\theta}{r^2} = \frac{\sin\theta}{r}$$

and $$\frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r}$$

Now $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\cdot\frac{\partial u}{\partial \theta}$$

$$\frac{\partial^2 u}{\partial x^2} = \cos\theta\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right) + \frac{\partial u}{\partial r}\cdot\frac{\partial}{\partial x}(\cos\theta) - \frac{1}{r}\cdot\frac{\partial u}{\partial \theta}\cdot\frac{\partial}{\partial x}(\sin\theta)$$

$$-\sin\theta\frac{\partial u}{\partial \theta}\cdot\frac{\partial}{\partial x}\left(\frac{1}{r}\right) - \frac{\sin\theta}{r}\cdot\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \theta}\right)$$

$$= \cos\theta\cdot\frac{\partial^2 u}{\partial r^2}\cos\theta + \frac{\partial u}{\partial r}(-\sin\theta)\cdot\left(-\frac{\sin\theta}{r}\right)$$

$$-\frac{1}{r}\cdot\frac{\partial u}{\partial \theta}(\cos\theta)\left(-\frac{\sin\theta}{r}\right) - \sin\theta\frac{\partial u}{\partial \theta}\left(-\frac{1}{r^2}\right)\cos\theta$$

$$-\frac{\sin\theta}{r}\cdot\frac{\partial^2 u}{\partial \theta^2}\cdot\left(-\frac{\sin\theta}{r}\right)$$

$$= \cos^2\theta\cdot\frac{\partial^2 u}{\partial r^2} + \frac{\sin^2\theta}{r^2}\cdot\frac{\partial^2 u}{\partial \theta^2} + \frac{\sin^2\theta}{r}\cdot\frac{\partial u}{\partial r}$$

$$+ \frac{2\sin\theta\cos\theta}{r^2}\cdot\frac{\partial u}{\partial \theta} \quad ...(1)$$

Scanned by CamScanner

[11] Prove that $u = (x^2 + y^2 + z^2)^{-1/2}$, $(x, y, z) \neq (0, 0, 0)$ is harmonic function of x, y, z.

Solution : Let $\sqrt{x^2 + y^2 + z^2} = r$

$$\therefore \quad u = \frac{1}{r}, \quad \frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}$$

$$\therefore \quad \frac{\partial u}{\partial x} = -\frac{1}{r^2} \cdot \frac{x}{r} = -\frac{x}{r^3}$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = \frac{3x}{r^4} \cdot \frac{x}{r} - \frac{1}{r^3} = \frac{3x^2}{r^5} - \frac{1}{r^3}$$

Similarly, $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{3x^2}{r^5} - \dfrac{1}{r^3}$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial z^2} = \frac{3z^2}{r^5} - \frac{1}{r^3}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{3(x^2 + y^2 + z^2)}{r^5} - \frac{3}{r^3}$$

$$= \frac{3r^2}{r^5} - \frac{3}{r^3} = 0$$

$\therefore$ u is harmonic function of x, y, z.

[12] Discuss Whether f_{xy} and f_{yx} for the following functions are equal or not.

(i) $f(x, y) = \log \sqrt{x^2 + y^2} + \tan^{-1} \dfrac{y}{x}$, $(x, y) \neq (0, 0)$

$= 0$, $(x, y) = (0, 0)$.

Solution : Here, $f(x, y) = \dfrac{1}{2}(x^2 + y^2) + \tan^{-1} \dfrac{y}{x}$

Scanned by CamScanner

$$\Rightarrow \quad f_x(x, y) = \frac{1}{2}\cdot\frac{2xy}{x^2+y^2} + \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = \frac{x-y}{x^2+y^2}$$

$$\Rightarrow \quad f_{xy}(x, y) = \frac{(x^2+y^2)(-1)-(x-y)\,2y}{(x^2+y^2)^2}$$

$$= \frac{y^2-2xy-x^2}{(x^2+y^2)^2} \quad \text{...(1)}$$

Now, $f_y(x, y) = \frac{1}{2}\cdot\frac{2y}{(x^2+y^2)} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x+y}{x^2+y^2}$

$$\Rightarrow \quad f_{yx}(x, y) = \frac{(x^2+y^2)(1)-(x+y)(2y)}{(x^2+y^2)^2}$$

$$= \frac{y^2-2xy-x^2}{(x^2+y^2)^2} \quad \text{...(2)}$$

$\therefore$ (1) and (2) $\Rightarrow f_{xy} = f_{yx}$

(ii) $\boldsymbol{F(x, y) = f(x + cy) + g(x - cy)}$

Solution : Here, $F(x, y) = f(x + cy) + g(x - cy)$

$\Rightarrow \quad F_x(x, y) = f'(x, cy) - g'(x - cy)$

$\quad F_{xy}(x, y) = cF''(x + cy) - cg''(x - cy)$ ...(1)

Now $F_y(x, y) = cf'(x + cy) - cg'(x - cy)$

$\Rightarrow \quad F_{yx}(x, y) = cf''(x + cy) - cg''(x - cy)$ ...(2)

$\therefore$ Results (1) and (2) $\Rightarrow f_{xy} = f_{yx}$

Scanned by CamScanner

2.10 Implicit function :

(a) The variable y is defined by a function of x by an relation $f(x, y) = c$, then y is called an implicit function of x.

Since $f(x, y) = c$ is a relation of variables x and y.

$$\therefore \quad f(x, y) = c \Rightarrow \frac{df}{dx} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow f_x + f_y \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{f_x}{f_y}, \text{ if } f_y \neq 0 \qquad ...(1)$$

$\therefore$ Differentiating both sides with respect to x,

$$\frac{d^2y}{dx^2} = -\frac{f_y\left[f_{xx} + f_{xy}\frac{dy}{dx}\right] - f_x\left[f_{yy}\frac{dy}{dx} + f_{yx}\right]}{[f_y]^2}$$

$$= -\frac{f_y\left[f_{xx} + f_{xy}\left(-\frac{f_x}{f_y}\right)\right] - f_x\left[f_{yy}\left(-\frac{f_x}{f_y}\right) + f_{yx}\right]}{[f_y]^2}$$

$$= -\frac{f_{xx}(f_y)^2 - 2f_x f_y f_{xy} + f_{yy}(f_x)^2}{(f_y)^3} \qquad [\because f_{yx} = f_{xy}]$$

(b) Let $u = f(x_1, x_2, x_3, \ldots\ldots, x_n)$ and $v = g(x_1, x_2, x_3, \ldots\ldots, x_n)$ are two differentiable functions of $x_1, x_2, \ldots\ldots, x_n$ defined on their common domain. Let they can be expressed by implicit functions as

$$F(u, v, x_1, x_2, \ldots\ldots, x_n) = 0 \qquad ...(1)$$

$$G(u, v, x_1, x_2, \ldots\ldots, x_n) = 0 \qquad ...(2)$$

Applying chain rule for equations (1) and (2)

$$\frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial x_i} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial x_i} = 0 \qquad ...(3)$$

Scanned by CamScanner

[2] **If $f(x, y) = 0$ and $g(y, z) = 0$, then show that**

$$f_y g_z \frac{dz}{dx} = f_x g_y .$$

Solution : Since $f(x, y) = 0 \Rightarrow \frac{dy}{dx} = -\frac{f_x}{f_y}$ ...(1)

and $g(y, z) = 0 \Rightarrow \frac{dz}{dy} = -\frac{g_y}{g_z}$ ...(2)

$\therefore$ Results (1) and (2) $\Rightarrow$

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = \left(-\frac{g_y}{g_z}\right)\left(-\frac{f_x}{f_y}\right)$$

$$f_y g_z \frac{dz}{dx} = f_x g_y$$

[3] **If $F(x, y, u, v) \equiv x^3 + y^3 + u^3 + 2v^3 - 5 = 0$ and $G(x, y, u, v) \equiv 2x^3 - y^3 + 3u^3 - v^3 - 7 = 0$, find**

$$\frac{\partial u}{\partial x}, \frac{\partial x}{\partial u}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 x}{\partial u^2}.$$

Solution : Here $F(x, y, u, v) = x^3 + y^3 + u^3 + 2v^3 - 5$

$G(x, y, u, v) = 2x^3 - y^3 + 3u^3 - v^3 - 7$

$\Rightarrow$ $F_x = 3x^2, F_y = 3y^2, F_u = 3u^2, F_v = 6v^2;$

$G_x = 6x^2, G_y = -3y^2, G_u = 9u^2, G_v = -3v^2$

$$\therefore \quad \frac{\partial u}{\partial x} = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} 3x^2 & 6v^2 \\ 6x^2 & -3v^2 \end{vmatrix}}{\begin{vmatrix} 3u^2 & 6v^2 \\ 9u^2 & -3v^2 \end{vmatrix}}$$

$$= -\frac{-45x^2v^2}{-63u^2v^2} = -\frac{5x^2}{7u^2}$$

Scanned by CamScanner

[4] If $F(x, y, r, \theta) \equiv x - r\cos\theta = 0$ and $G(x, y, r, \theta) \equiv y - r\sin\theta = 0$, then prove that

$$\frac{\partial r}{\partial x} = \cos\theta, \quad \frac{\partial r}{\partial y} = \sin\theta, \quad \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r} \text{ and } \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}.$$

Solution : Here $F(x, y, r, \theta) \equiv x - r\cos\theta$ and

$$G(x, y, r, \theta) \equiv y - r\sin\theta$$

$\therefore$ $F_x = 1,\ F_y = 0,\ F_r = -\cos\theta,\ F_\theta = r\sin\theta$

$G_x = 0,\ G_y = 1,\ G_r = -\sin\theta,\ G_\theta = -r\cos\theta$

$$\therefore \quad \frac{\partial r}{\partial x} = -\frac{\begin{vmatrix} F_x & F_\theta \\ G_x & G_\theta \end{vmatrix}}{\begin{vmatrix} F_r & F_\theta \\ G_r & G_\theta \end{vmatrix}} = -\frac{\begin{vmatrix} 1 & r\sin\theta \\ 0 & -r\cos\theta \end{vmatrix}}{\begin{vmatrix} -\cos\theta & r\sin\theta \\ -\sin\theta & -r\cos\theta \end{vmatrix}}$$

$$= -\frac{-r\cos\theta}{r} = \cos\theta$$

$$\frac{\partial r}{\partial y} = -\frac{\partial(F,G)/\partial(y,\theta)}{\partial(F,G)/\partial(r,\theta)} = -\frac{\begin{vmatrix} 0 & r\sin\theta \\ 1 & -r\cos\theta \end{vmatrix}}{r}$$

$$= -\frac{-r\sin\theta}{r} = \sin\theta$$

$$\frac{\partial \theta}{\partial x} = -\frac{\partial(F,G)/\partial(r,x)}{\partial(F,G)/\partial(r,\theta)} = -\frac{\begin{vmatrix} -\cos\theta & 1 \\ -\sin\theta & 0 \end{vmatrix}}{r} = -\frac{\sin\theta}{r}$$

$$\text{and} \quad \frac{\partial \theta}{\partial y} = -\frac{\partial(F,G)/\partial(r,y)}{\partial(F,G)/\partial(r,\theta)} = -\frac{\begin{vmatrix} -\cos\theta & 0 \\ -\sin\theta & 1 \end{vmatrix}}{r} = \frac{\cos\theta}{r}$$

Scanned by CamScanner

Unit – III

Scanned by CamScanner

Unit : 3

Chapter 3

Differentiability of function of several variables-II

3.1 : Total Differential :

If $z = f(x, y)$ is a real function defined on non-empty open set $E \subset R^2$, which is differentiable at point $(x, y) \in E$ then

$$\delta z = f_x\, \delta x + f_y\, \delta y + \varepsilon \varrho$$

where $\varepsilon \to 0$ as $\varrho \to 0$ ...(1)

Here principal part $f_x \delta x + f_y\, \delta y$ of δz is called total differential of z and is denoted by dz.

Thus $dz = f_x\, \delta x + f_y\, \delta y$ ...(2)

But x and y are independent variables.

$\delta x = dx,\ \delta y = dy$ ...(3)

$\therefore$ Results (2) and (3) $\Rightarrow dz = f_x dx + f_y dy$ ...(4)

Theorem 1 :

If a function $f : (x, y) \to z = f(x, y)$ possess continuous partial derivatives in its domain and if the functions $\phi : t \to x = \phi(t)$ and $\psi : t \to y = \psi(t)$ possess continuous derivatives in their domain $[a, b]$, then $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial z}{\partial y} \cdot \dfrac{dy}{dt}$.

Proof : Let $t,\ t + \delta t \in [a, b]$

δt is the increment of t.

$\Rightarrow$ $\exists\, \delta x$ and δy which are respectively increments of x and y

$\Rightarrow$ $\exists\, \delta z$ which is the increment of z.

Scanned by CamScanner

$\therefore \quad \delta z = f(x+\delta x, y+\delta y) - f(x, y)$

$$= f(x+\delta x, y+\delta y) - f(x, y+\delta y) + f(x, y+\delta y) - f(x, y) \quad ...(1)$$

But f_x and f_y exist at point $(x, y) \Rightarrow$

by L. M. V. theorem

$$f(x+\delta x, y+\delta y) - f(x, y+\delta y) = f_x(x+\theta_1 \delta x, y+\delta y)\delta x$$

and $f(x, y+\delta y) - f(x, y) = f_y(x, y+\theta_2 \delta y)\,\delta y$

where $\theta_1, \theta_2 \in (0, 1)$ ...(2)

$\therefore$ Results (1) and (2) $\Rightarrow$

$$\delta z = f_x(x+\theta_1 \delta x, y+\delta y)\delta x + f_x(x, y+\theta_2 \delta y)\delta y$$

where $\theta_1, \theta_2 \in (0, 1)$

$\therefore$ If $\delta t \neq 0$, then

$$\frac{\delta z}{\delta t} = f(x+\theta_1\delta x, y+\delta y)\frac{\delta x}{\delta t} + f_y(x, y+\theta_2\delta y)\frac{\delta y}{\delta t}$$

where $\theta_1, \theta_2 \in (0, 1)$ ...(3)

Let $\delta t \to 0$ then $\delta x \to 0, \ \delta y \to 0$

Now f_x and f_y are continuous at point (x, y)

$$\therefore \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} f_x(x+\theta_1 \delta x, y+\delta y) = f_x(x, y) = \frac{\partial z}{\partial x} \quad ...(4)$$

and $$\lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} f_y(x, y+\theta_2\delta y) = f_y(x, y) = \frac{\partial z}{\partial y} \quad ...(5)$$

Now from result (3)

$$\lim_{\delta t \to 0} \frac{\partial z}{\partial t} = \lim_{\delta t \to 0} f_x(x+\theta_1 \delta x, y+\delta y)\frac{\delta x}{\delta t}$$

$$+ \lim_{\delta t \to 0} f_y(x, y+\theta_2 \delta y)\frac{\delta y}{\delta t}$$

Scanned by CamScanner

$$= \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} f_x(x+\theta_1 \delta x,\ y+\delta y) \lim_{\delta t \to 0} \frac{\delta x}{\delta t}$$

$$+ \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} f_y(x,\ y+\theta_2 \delta y) \lim_{\delta t \to 0} \frac{\delta y}{\delta t}$$

By results (4) and (5),

$\therefore \quad \frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$ which is the required result.

Theorem 2 :

If function $f : x \to z = f(x)$ possess continuous partial derivative in its domain $E \subset R^n$, and $\phi_i : t \to \phi_i(t) = x_i$ possess continuous derivatives in their domain $[a, b]$, then

$$\frac{dz}{dt} = \sum_{i=1}^{n} \frac{\partial z}{\partial x_i} \cdot \frac{dx_i}{dt},\ i = 1, 2, 3, \ldots, n.$$

Proof : Since $z = f(x) = f(x_1, x_2, \ldots, x_n)$

and $x_i = \phi_i(t),\ i = 1, 2, 3, \ldots, n$

$\therefore \quad z = f(\phi_1(t), \phi_2(t), \ldots, \phi_n(t)) = f(p(t))$

$= G(t)$ Take $p \in E$

$$\therefore \quad G'(t) = \lim_{h \to 0} \frac{G(t+h) - G(t)}{h}$$

Now let $G(t+h) = f(\phi_1(t+h), \phi_2(t+h), \ldots, \phi_n(t+h))$

$= f(B)$

and $\quad G(t) = f(\phi_1(t), \phi_2(t), \ldots, \phi_n(t)) = f(A)$

$G(t+h) - G(t) = f(B) - f(A)$

Since f is differentiable function on E and each x_i and its derivatives are continuous functions on $[a, b]$.

∴ By Lagrange's mean value theorem,

$$G(t+h) - G(t) = f(B) - f(A)$$

$$= f'(A+\theta(B-A)) \cdot (B-A),\ \theta \in (0,1)$$

$$= \nabla f(A+\theta(B-A)) \cdot (B-A),\ \theta \in (0,1)$$

$$G'(t) = \lim_{h \to 0} \frac{\nabla f(A+\theta(B-A)) \cdot (B-A)}{h}$$

$$= \lim_{h \to 0} \nabla f(A+\theta(B-A)) \cdot \lim_{h \to 0} \frac{B-A}{h}$$

As $h \to 0 \Rightarrow B \to A$

$$= \nabla(f(A)) \lim_{h \to 0} \frac{P(t+h) - P(t)}{h}$$

$$= \nabla f(P(t))\, P'(t)$$

$$= \nabla f(x) \cdot (\phi_1'(t), \phi_2'(t), \ldots\ldots, \phi_n'(t))$$

$$= \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots\ldots, \frac{\partial f}{\partial x_n}\right) \cdot \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \ldots\ldots, \frac{dx_n}{dt}\right)$$

$$= \sum_{i=1}^{n} \frac{\partial z}{\partial x_i} \cdot \frac{dx_i}{dt}$$

Scanned by CamScanner

Theorem 1 : Euler's theorem on homogeneous function :

f is a differentiable homogeneous function of degree m invariables x_i, $i = 1, 2, 3, \ldots\ldots, n$ defined on the non-empty open set $E \subset R^n \Leftrightarrow \sum_{i=1}^{n} x_i \dfrac{\partial f}{\partial x_i} = m f(x)$.

Proof : Since f is a homogeneous function of degree m in variables x_i, $i = 1, 2, 3, \ldots\ldots, n$.

$\therefore$ Let $u = f(x_1, x_2, x_3, \ldots\ldots, x_n)$

$$\therefore \quad u = x_1^m F\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \ldots\ldots, \frac{x_n}{x_1}\right)$$

Put $\dfrac{x_i}{x_1} = v_i$, $i = 2, 3, 4, \ldots\ldots, n$.

$$\therefore \quad u = x_1^m F(v_2, v_3, v_4, \ldots\ldots, v_n)$$

$$\therefore \quad \frac{\partial u}{\partial x_1} = m x_1^{m-1} F + x_1^m \sum_{i=2}^{m} \frac{\partial F}{\partial v_i} \cdot \frac{\partial v_i}{\partial x_1}$$

$$= m x_1^{m-1} F + x_1^m \sum_{i=2}^{m} \frac{\partial F}{\partial v_i} \cdot \left(-\frac{x_i}{x_1^2}\right) \qquad \left(\because v_i = \frac{x_i}{x_1}\right)$$

$$= m x_1^{m-1} F - x_1^{m-2} \sum_{i=2}^{m} x_i \frac{\partial F}{\partial v_i} \qquad \ldots(1)$$

$$\frac{\partial u}{\partial x_i} = x_1^m \frac{\partial F}{\partial v_i} \cdot \frac{\partial v_i}{\partial x_i} = x_1^m \cdot \frac{\partial F}{\partial v_i} \cdot \frac{1}{x_1}$$

$$= x_1^{m-1} \cdot \frac{\partial F}{\partial v_i} \qquad \ldots(2)$$

Scanned by CamScanner

$$\therefore \quad \sum_{i=1}^{m} x_i \frac{\partial u}{\partial x_i} = x_1 \frac{\partial u}{\partial x_1} + \sum_{i=2}^{m} x_i \frac{\partial u}{\partial x_i}$$

$$= mx_1^m F(v_2, v_3, \ldots\ldots, v_n)$$

$$- x_1^{m-1} \sum_{i=2}^{m} x_i \frac{\partial F}{\partial v_i} + x_1^{m-1} \sum_{i=2}^{m} x_i \frac{\partial F}{\partial v_i}$$

$$= mx_1^m F(v_2, v_3, \ldots\ldots, v_n)$$

$$= mf(x_1, x_2, \ldots\ldots, x_n)$$

Part II : Suppose that $\sum_{i=1}^{m} x_i \frac{\partial f}{\partial x_i} = mf(x_1, x_2, \ldots\ldots, x_n)$

Let $v = f(tx_1, tx_2, tx_3, \ldots\ldots, tx_n)$, $t > 0$.

Put $tx_i = X_i$, $i = 1, 2, \ldots\ldots, n$.

$$\therefore \quad v = f(X_1, X_2, \ldots\ldots, X_n) \text{ and } \frac{dX_i}{dt} = x_i$$

$$\therefore \quad \frac{dv}{dt} = \sum_{1}^{n} \frac{\partial f}{\partial X_i} \cdot \frac{dX_i}{dt} = \sum_{1}^{n} \frac{\partial f}{\partial X_i} \cdot x_i$$

$$\therefore \quad t\frac{dv}{dt} = \sum_{1}^{n} tx_i \frac{\partial f}{\partial X_i} = \sum_{1}^{n} X_i \frac{\partial f}{\partial X_i} \quad \ldots(1)$$

Now since $\sum_{1}^{n} x_i \cdot \frac{\partial f}{\partial x_i} = mf$

$$\Rightarrow \quad \sum_{1}^{n} X_i \frac{\partial f}{\partial X_i} = m\, f(X_1, X_2, \ldots\ldots, X_n) = mv \quad \ldots(2)$$

$\therefore$ Results (1) and (2), $t \frac{dv}{v} = mv$

Scanned by CamScanner

$$\Rightarrow \quad \frac{dv}{v} = m\frac{dt}{t}$$

$$\Rightarrow \quad \log v = m \log t + \log c$$

$$\Rightarrow \quad \log \frac{v}{c} = \log t^m$$

$$\Rightarrow \quad v = ct^m \qquad \text{...(3)}$$

If $t = 1$, then $v = u \Rightarrow c = u$

Result (3) and (4) $\Rightarrow v = ut^m$

$\therefore \quad f(tx_1, tx_2, \ldots\ldots, tx_n) = t^m f(x_1, x_2, \ldots\ldots, x_n)$

$\therefore \quad f$ is homogeneous function of degree m in variables.

x_i, $i = 1, 2, \ldots\ldots, n$.

Corollary 1 : f is a differentiable homogeneous function of two variables x and y of degree $m \Leftrightarrow$

$$x f_x + y f_y = mf(x, y).$$

Proof : Part I : Suppose that $f(x, y)$ is a homogeneous function of x and y of degree m

$\therefore$ Let $f(x, y) = x^m F(y/x)$

$= x^m F(v)$ (Putting $y/x = v$)

Since f is differentiable function of x and y.

$\therefore$ F is also differential function of x and y.

$$\therefore \quad \frac{\partial f}{\partial x} = mx^{m-1} F(v) + x^m \cdot \frac{\partial F}{\partial v} \cdot \frac{dv}{dx}, \; \frac{dv}{dx} = -\frac{y}{x^2}$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} = mx^{m-1} F(v) - x^{m-2} \cdot y\frac{\partial F}{\partial v} \qquad \text{...(1)}$$

$$\text{and} \quad \frac{\partial f}{\partial y} = x^m \cdot \frac{\partial F}{\partial v} \cdot \frac{dv}{dy}, \; \frac{dv}{dy} = \frac{1}{x}$$

$$\Rightarrow \quad \frac{\partial f}{\partial y} = x^{m-1} \cdot \frac{\partial F}{\partial v} \qquad \text{...(2)}$$

Scanned by CamScanner

$\therefore$ Results (1) and (2) $\Rightarrow$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = m\,x^m F(v) - x^{m-1}\,y\frac{\partial F}{\partial v} + x^{m-1}\,y\frac{\partial F}{\partial v}$$

$$= m\,x^m F(v)$$

$$= m\,f(x, y)$$

Part II : Suppose that for a differentiable function $f(x, y)$.

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = m\,f(x, y) \quad \text{...(1)}$$

Let $u = f(x, y)$ and $v = f(tx, ty)$, $t > 0$

Put $tx = X$, $ty = Y$

$v = f(X, Y)$

$$\therefore \quad \frac{dv}{dt} = \frac{\partial f}{\partial X}\cdot\frac{dX}{dt} + \frac{\partial f}{\partial Y}\cdot\frac{dY}{dt}$$

$$= x\frac{\partial f}{\partial X} + y\frac{\partial f}{\partial Y} \qquad \left(\because \frac{dX}{dt} = x,\ \frac{dY}{dt} = y\right)$$

$$\therefore \quad t\cdot\frac{dv}{dt} = tx\frac{\partial f}{\partial X} + ty\frac{\partial f}{\partial Y}$$

$$= X\frac{\partial f}{\partial X} + Y\frac{\partial f}{\partial Y}$$

$$= m\,f(X, Y) \qquad \text{by result (1)}$$

$$= mv.$$

$$\therefore \quad \frac{dv}{v} = m\frac{dt}{t} \Rightarrow \log v = m\log t + \log c$$

$$\Rightarrow \log\frac{v}{c} = \log t^m$$

$$\Rightarrow v = ct^m \quad \text{...(2)}$$

Scanned by CamScanner

If $t = 1$, then $v = u$

$\therefore$ From result (2), $u = c \cdot 1^m = c$ ...(3)

$\therefore$ Results (2) and (3),

$$\Rightarrow v = t^m u$$

$\therefore$ $f(tx \,.\, ty) = t^m f(x, y)$

$\therefore$ The function $f(x, y)$ is the homogeneous function of x and y of degree m.

Corollary 2 : If $f(x, y)$ is a homogeneous function of x and y of degree m and if its second order partial derivatives exist, then

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = m(m-1) f(x, y)$$

Proof : f is a homogeneous function of x and y of degree m and its partial derivatives of second order exist.

$$\therefore \quad x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = m f(x, y) \quad \text{...(1)}$$

$\therefore$ Differentiating partially with respect to x and y respectively we get

$$\frac{\partial f}{\partial x} + x\frac{\partial^2 f}{\partial x^2} + y\frac{\partial^2 f}{\partial x \partial y} = m\frac{\partial f}{\partial x} \quad \text{...(2)}$$

$$\text{and } x\frac{\partial^2 f}{\partial y \partial x} + \frac{\partial f}{\partial y} + y\frac{\partial^2 f}{\partial y^2} = m\frac{\partial f}{\partial y} \quad \text{...(3)}$$

Multiply both sides of results (2) and (3) respectively by x and y and add the results and here

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Scanned by CamScanner

$$\therefore \quad x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\partial y} + y^2\frac{\partial^2 f}{\partial y^2}$$

$$= (m-1)\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right)$$

$$= m\,(m-1)\,f(x, y) \qquad \text{by result (1).}$$

Theorem 2 :

If $u = \phi(H)$ is a function of a homogeneous function $H = f(x, y)$ of degree m whose partial derivatives of second order exist, then

(i) $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = m\frac{F(u)}{F'(u)},\ F'(u) \neq 0$$

$$= G(u) \text{ say}$$

(ii) $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = G(u)\,[G'(u) - 1]$$

Where $H = f(x, y) = F(u) = \phi^{-1}(u)$

Proof : Let $u = \phi(f(x, y)) = \phi(H)$ say

$\therefore \quad \phi^{-1}(u) = F(u) = f(x, y) = H$ ...(1)

Since $H = f(x, y)$ is a homogeneous function of x and y of degree m whose partial derivatives exist.

$$\therefore \quad x\frac{\partial H}{\partial x} + y\frac{\partial H}{\partial y} = mH = mF(u) \qquad ...(2)$$

But $$\frac{\partial H}{\partial x} = \frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial x} = F'(u)\frac{\partial u}{\partial x}$$

and $$\frac{\partial H}{\partial y} = \frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial y} = F'(u)\frac{\partial u}{\partial y}$$

Scanned by CamScanner

$$\therefore \quad xF'(u)\frac{\partial u}{\partial x} + yF'(u)\frac{\partial u}{\partial y} = mF(u) \qquad \text{By result (2)}$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = m\frac{F(u)}{F'(u)}, \quad F'(u) \neq 0 \qquad \text{...(3)}$$

$$= G(u) \text{ say}$$

$\therefore$ Differentiating result (3) partially w. r. to x and y resp.

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = G'(u)\frac{\partial u}{\partial x} \qquad \text{...(4)}$$

$$\text{and} \quad x\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = G'(u)\frac{\partial u}{\partial y} \qquad \text{...(5)}$$

$$\text{Here} \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

Multiplying both sides of results (4) and (5) by x an y respectively and adding the results.

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$= G'(u)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$\Rightarrow \quad x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (G'(u) - 1)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$= G(u)\,(G'(u) - 1)$$

[by result (3)]

Scanned by CamScanner

3.6 Lagrange's method of undetermined multiplies, to detemine extreme values of a function of n variables.

Proof : Let u be a real differentiable function of n variables $x_1, x_2, x_3, \ldots\ldots, x_n$ say

$$u = \phi(x_1, x_2, x_3, \ldots\ldots, x_n)$$

whose extreme values are to be determined subject to $m\,(<n)$ conditions (equations)

$$f_i(x_1, x_2, x_3, \ldots\ldots, x_n) = 0, \quad i = 1, 2, 3, \ldots\ldots, m \qquad \ldots(2)$$

Where each f_i is differentiable.

So that only $n - m$ of n variables are independent.

For extreme values of u.

$$du = \sum_{i=1}^{n} \frac{\partial f_r}{\partial x_i} dx_i = \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i} dx_i = 0 \qquad \ldots(3)$$

and $$df_r = \sum_{i=1}^{n} \frac{\partial f_r}{\partial x_i} dx_i = 0 \ \ldots\ldots (r+3), \quad r = 1, 2, 3, \ldots\ldots; m$$

Multiply both sides of results (3), (4), (5), ..., $(m+3)$ respectively by $1, \lambda_1, \lambda_2, \ldots\ldots, \lambda_m$, and add all results by columns.

∴ We have the equation.

$$P_1 dx_1 + P_2 dx_2 + P_3 dx_3 + \ldots\ldots P_n dx_n = 0 \qquad \ldots(I)$$

Where $$P_i = \frac{\partial \phi}{\partial x_i} + \lambda_1 \frac{\partial f_1}{\partial x_i} + \lambda_1 \frac{\partial f_2}{\partial x_i} + \ldots\ldots + \lambda_m \frac{\partial f_m}{\partial x_i},$$

$$i = 1, 2, 3, \ldots\ldots, n \qquad \ldots(II)$$

Now $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots, \lambda_m$, used above are called undetermined multipliers and are at our choice. So we choose them so as to satisfy the following m equations

$$P_1 = 0, P_2 = 0, P_3 = 0, P_m = 0 \qquad \ldots(III)$$

∴ The equation (I) reduces to

$$P_{m+1} dx_{m+1} + P_{m+2} dxm_{m+2} + \ldots + P_n dx_n = 0 \qquad \ldots(IV)$$

Scanned by CamScanner

Since any $n - m$ variables of n variables $x_1, x_2, x_3, \ldots\ldots, x_n$ can be taken as independent values, let they be taken as $x_{m+1}, x_{m+2}, \ldots\ldots, x_n$, so from the equation (IV)

$$P_{m+1} = 0, \quad P_{m+2} = 0, \; \ldots\ldots, \; P_n = 0 \qquad \ldots(V)$$

Thus in virtue of choice of λ's, from equations (III) and (V),

$$P_1 = P_2 = P_3 = \ldots\ldots = P_n = 0$$

These n conditions (VI) along with the given m equations to determine m unknown multipliers $\lambda_1, \lambda_2, \ldots\ldots, \lambda_m$ and the values of n variables $x_1, x_2, \ldots\ldots, x_n$ for which extreme value of u may exist.

3.7 Illustrations :

[1] **Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition, $ax + by + cz - d = 0$, $a^2 + b^2 + c^2 \neq 0$, $d \neq 0$.**

Solution : Let $F(x, y, z) = (x^2 + y^2 + z^2) + \lambda(ax + by + cz - d)$

$$\Rightarrow \left.\begin{array}{l} P_1 = F_x = 2x + \lambda a = 0 \\ P_2 = F_y = 2y + \lambda b = 0 \\ P_3 = F_z = 2z + \lambda c = 0 \end{array}\right\} \Rightarrow \lambda = -\frac{2x}{a} = -\frac{2y}{b} = -\frac{2z}{c}$$

$$\Rightarrow \frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{x^2 + y^2 + z^2}{d} \qquad \ldots(1)$$

$$= \frac{ax + by + cz}{a^2 + b^2 + c^2} = \frac{d}{a^2 + b^2 + c^2} \qquad \ldots(2)$$

$\therefore$ Results (1) and (2) $\Rightarrow$

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{d}{a^2 + b^2 + c^2} = \frac{u}{d}, \text{ where } u = x^2 + y^2 + z^2$$

$$\Rightarrow \quad x = \frac{ad}{\Sigma a^2}, \; y = \frac{bd}{\Sigma a^2}, \; z = \frac{cd}{\Sigma a^2}, \; u = \frac{d}{a^2 + b^2 + c^2}$$

Now, $u = x^2 + y^2 + z^2$, $ax + by + cz - d = 0$

Scanned by CamScanner

$$\Rightarrow \quad \frac{\partial u}{\partial x} = 2x + 2z\frac{\partial z}{\partial x}, \; a + c\frac{\partial z}{\partial x} = 0, \; b + c\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = 2x + 2z\left(-\frac{a}{c}\right) = 2x - \frac{2az}{c}$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = 2 - \frac{2a\partial z}{c\partial x} = 2 - \frac{2a}{c}\left(-\frac{a}{c}\right) = 2 + \frac{2a^2}{c^2}.$$

Similarly, $\frac{\partial u^2}{\partial y^2} = 2 + 2\frac{b^2}{c^2}$

and $\quad \frac{\partial^2 u}{\partial x \partial y} = -\frac{2a\partial z}{c\partial y} = \frac{2a}{c}\left(-\frac{b}{c}\right) = \frac{2ab}{c^2}$

$\therefore \quad r > 0$ and $rt - s^2 = \left(2 + \frac{2a^2}{c^2}\right)\left(2 + \frac{db^2}{c^2}\right) - \left(\frac{2ab}{c^2}\right)^2$

$$= 4\left(1 + \frac{a^2}{c^2} + \frac{b^2}{c^2}\right) > 0$$

$\therefore \quad u = \frac{d^2}{\Sigma a^2}$ is minimum at point $\left(\frac{ad}{\Sigma a^2}, \frac{bd}{\Sigma a^2}, \frac{cd}{\Sigma a^2}\right)$.

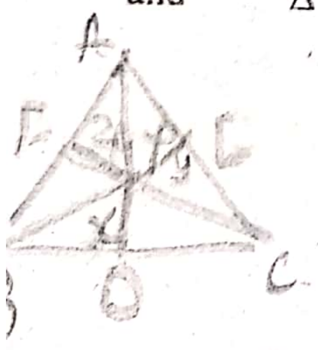
[2] Find a point within a triangle such that the sum of squares of its distances from the sides of the triangle is a minimum.

Solution : Let P be the point in ΔABC.

Draw $\overline{PD} \perp \overline{BC}$, $\overline{PE} \perp \overline{CA}$, $\overline{PF} \perp \overline{AB}$ Join $\overline{PA}$, $\overline{PB}$, $\overline{PC}$

Let $PD = x$, $PE = y$, $PF = z$. Then $u = x^2 + y^2 + z^2$

and $\quad \Delta = \Delta BPC + \Delta CPA + \Delta APB$ ...(1)



$$= \frac{1}{2}ax + \frac{1}{2}by + \frac{1}{2}cz$$

Scanned by CamScanner

$$\Rightarrow \quad ax + by + cz = 2\Delta \qquad ...(2)$$

Let $F(x, y, z) = x^2 + y^2 + z^2 + \lambda(ax + by + cz - 2\Delta)$ ...(3)

$$\Rightarrow \quad P_1 = F_x = 2x + a\lambda = 0 \qquad ...(4)$$

$$P_2 = F_y = 2y + b\lambda = 0 \qquad ...(5)$$

$$P_3 = F_z = 2z + c\lambda = 0 \qquad ...(6)$$

Multiply both sides of (4), (5), (6) by x, y, z respectively and add the results by columns :

$$\therefore \quad 2(x^2 + y^2 + z^2) + \lambda(ax + by + cz) = 0$$

$$\Rightarrow 2u + 2\lambda\Delta = 0 \qquad \text{by results (1) and (2)}$$

$$\Rightarrow \quad \lambda = -\frac{u}{\Delta} \qquad ...(7)$$

$\therefore$ From results (4), (5), (6) and (7)

$$x = -\frac{\lambda a}{2} = \frac{ua}{2\Delta}, \; y = \frac{ub}{2\Delta}, \; z = \frac{uc}{2\Delta} \qquad ...(8)$$

$$\therefore \quad u = x^2 + y^2 + z^2 = \frac{u^2}{4\Delta^2}(a^2 + b^2 + c^2)$$

$$\Rightarrow \quad u = \frac{4\Delta^2}{a^2 + b^2 + c^2} \qquad ...(9)$$

$\therefore$ Results (8), (9)

$$\Rightarrow \quad x = \frac{2\Delta a}{\Sigma u^2}, \; y = \frac{2\Delta b}{\Sigma u^2}, \; z = \frac{2\Delta c}{\Sigma u^2}$$

Since $u = x^2 + y^2 + z^2$ and $ax + by + cz - 2\Delta = 0$

Reader can verify as example (1) that

$$\frac{\partial^2 u}{\partial x^2} = 2 + \frac{2a^2}{c^2}, \; \frac{\partial u^2}{\partial x \partial y} = \frac{2ab}{c^2}, \; \frac{\partial^2 u}{\partial y^2} = 2 + 2\frac{b^2}{c^2}$$

$$\Rightarrow \quad r > 0 \text{ and } rt - s^2 = 4\left(1 + \frac{a^2}{c^2} + \frac{b^2}{c^2}\right) > 0$$

Scanned by CamScanner

$\therefore \quad \dfrac{4\Delta^2}{a^2+b^2+c^2}$ is minimum value of u at point

$$\left(\frac{2\Delta a}{a^2+b^2+c^2}, \frac{2\Delta b}{a^2+b^2+c^2}, \frac{2\Delta c}{a^2+b^2+c^2}\right)$$ which is the centroid of $\Delta\, ABC$ that can be verified by the reader.

[3] **Prove that the extreme value of $u = ax^2 + by^2 + cz^2$ subject to the conditions $x^2 + y^2 + z^2 = 1$, $lx + my + nz = 0$ is given by $\dfrac{l^2}{a^2-u} + \dfrac{m^2}{b^2-u} + \dfrac{u^2}{c^2-u} = 0$.**

Solution : Let $F(x, y, z) = (a^2x^2 + b^2y^2 + c^2z^2) + \lambda(x^2 + y^2 + z^2 - 1)$
$\qquad + \mu(lx + my + nz)$ ...(1)

$\therefore \quad P_1 = F_x = 2a^2x + 2\lambda x + l\mu = 0$ ...(2)

$P_2 = F_y = 2b^2y + 2\lambda y + m\mu = 0$ ...(3)

$P_3 = F_z = 2c^2z + 2\lambda z + n\mu = 0$ ...(4)

Multiply both sides of (2), (3) and (4) by x, y and z respectively and add the results by columns.

$\therefore$ We have $2u + 2\lambda \cdot 1 + \mu \cdot 0 = 0 \Rightarrow \lambda = -u$ ...(5)

$\therefore$ Results (2), (3), (4) and (5) $\Rightarrow$

$$x = -\frac{l\mu}{2(a^2-u)}, \quad y = -\frac{m\mu}{2(b^2-u)}, \quad z = -\frac{u\mu}{2(c^2-u)}$$

Substituting these values of x, y, z in equation $lx + my + nz = 0$, we have

$$-\frac{\mu}{2}\left(\frac{l^2}{a^2-u} + \frac{m^2}{b^2-u} + \frac{u^2}{c^2-u}\right) = 0$$

$$\therefore \quad \frac{l^2}{a^2-u} + \frac{m^2}{b^2-u} + \frac{u^2}{c^2-u} = 0 \qquad (\because \mu \neq 0)$$

which is the required result.

Scanned by CamScanner

Unit – IV

Scanned by CamScanner

Unit : 4

Chapter 4

Applications of partial derivatives

4.1 : Expansion of a function $f(x, y)$:

In F.Y. B.A/ B. Sc. classes we have studied the Taylor's theorem and Maclaurin's theorem for the expansion the function of one variable. Now we shall study them for the expansions of the functions of two variables.

Taylor's theorem :

Let f be a function defined on domain $E \subset R^2$. If function f possess its continuous partial derivatives upto n^{th} order in a nbhd. N of a point $(x, y) \in E$ and if $(x + h, y + k) \in N$, then there exists $\theta \in (0, 1)$ such that

$$f(x + h, y + k) = f(x, y) + \frac{1}{r!}\sum_{r=1}^{n-1}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^r f(x, y)$$

$$+ \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x + \theta h, y + \theta k)$$

Proof : Define a function $g(t) = f(x + ht, y + kt)$ where t is an auxiliary variable.

$\therefore$ Using Maclaurin's theorem for one variable t.

$$g(t) = g(0) + \sum_{r=1}^{n-1}\frac{t^r}{r!}g^r(0) + \frac{t^n}{n!}g^n(\theta t),\ \theta \in (0, 1) \qquad ...(1)$$

Scanned by CamScanner

Now since x and y are independent variables of t and

$g(t) = f(x + ht, y + kt) \Rightarrow$ by chain rule

$$g^n(t) = \frac{d^n}{dt^n}(g(t)) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n g(t)$$

$$= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x+ht,\ y+kt)$$

$$\Rightarrow \quad g^n(0) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x, y) \quad ...(2)$$

$\therefore$ Results (1) and (2) $\Rightarrow$

$\therefore$ $f(x + ht, y + ht) = g(t)$

$$= g(0) + \sum_{r=1}^{n-1} \frac{t^r}{r!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^r f(x, y)$$

$$+ \frac{t^n}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n g(\theta t),\ \theta \in (0, 1) \quad ...(3)$$

Putting $t = 1$ result (3)

$$f(x + h, y + k) = f(x, y) + \sum_{r=1}^{n-1} \frac{1}{r!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^r f(x, y)$$

$$+ \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x+\theta h,\ y+\theta k),\ \theta \in (0, 1) \quad ...(4)$$

If we replace $x + h$ by x and $y + k$ by y, x by a and y by b, then reduces to

Scanned by CamScanner

$$f(x, y) = f(a, b) + \sum_{r=1}^{n-1} \frac{1}{r!}\left((x-a)\frac{\partial}{\partial y} + (y-b)\frac{\partial}{\partial y}\right)^r f(a, b)$$

$$+ \frac{1}{n!}\left((x-a)\frac{\partial}{\partial y} + (y-b)\frac{\partial}{\partial y}\right)^n f(a+\theta(x-a),\ b+\theta(y-b)),$$

$$\theta \in (0,1) \quad ...(5)$$

Maclaurin's theorem :

Let f be a real function defined on the domain $E \subset R^2$. If the function f possess its continuous partial derivatives upto n[th] order in a nbhd $N \subset E$ of the point (0, 0) and if $(x, y) \in N$, then there exists $\theta \in (0, 1)$ such that

$$f(x, y) = f(0, 0) + \sum_{r=1}^{n-1} \frac{1}{r!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^r f(0, 0)$$

$$+ \frac{1}{n!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^n f(\theta x, \theta y)$$

In Taylor's formula (4), put $x = y = 0$ and replace h by x and k by y and hence the result.

L. Hospital rule for the function of two variables :

Let $f : E(\subset R \times R) \to R$ and $g : E(\subset R \times R) \to R$ be the functions and $(a, b) \in E$, if $h, k \in R^+$ and if

(i) $f(a, b) = g(a, b) = 0$

(ii) Function f and g are continuous in E.

(iii) Functions f and g are partially differential function for x and y in E and if $g_x \neq 0$, $g_y \neq 0$, and if $(x, y) \to (a, b)$ along st. line $y = \lambda x$, then,

$$\lim_{(x,y)\to(a,b)} \frac{f(x, y)}{g(x, y)} = \frac{f_x(a,b) + \lambda f_y(a,b)}{g_x(a,b) + \lambda g_y(a,b)}$$

Scanned by CamScanner

Proof : Let $\dfrac{f(x, y)}{g(x, y)} = \dfrac{f(a+h, b+k)}{g(a+h, b+k)}$

$$= \frac{f(a,b) + hf_x(a+\theta_1 h, b+\theta_1 k) + kf_y(a+\theta_2 h, b+\theta_2 k)}{g(a,b) + hg_x(a+\theta_3 h, b+\theta_3 k) + kg_y(a+\theta_4 h, b+\theta_4 k)}$$

$$, \theta_1, \theta_2, \theta_3, \theta_4 \in (0,1)$$

But $f(a, b) = g(a, b) = 0$

$$\therefore \quad \frac{f(x, y)}{g(x, y)} = \frac{hf_x(a+\theta_1 h, b+\theta_1 k) + kf_y(a+\theta_1 h, b+\theta_1 k)}{hg_x(a+\theta_2 h, b+\theta_2 k) + kg_y(a+\theta_2 h, b+\theta_2 h)}$$

But $(x, y) \to (a, b)$ to along the line $y = \lambda x$, then

$$= \frac{f_x(a,b) + \lambda f_y(a,b)}{g_x(a,b) + \lambda f_y(a,b)}$$

4.2 Illustrations :

1. Find first three terms in the expansion of $f(x, y) = e^{ax} \sin by$ in powers of x and y.

Solution : Here $f(0, 0) = 0$.

$$\frac{\partial}{\partial x} f(x, y) = ae^{ax} \sin by \quad \Rightarrow \quad \frac{\partial}{\partial x} f(0, 0) = 0$$

$$\frac{\partial}{\partial y} f(x, y) = be^{ax} \cos by \quad \Rightarrow \quad \frac{\partial}{\partial y} f(0, 0) = b$$

$$\frac{\partial^2}{\partial x^2} f(x, y) = a^2 e^{ax} \sin by \quad \Rightarrow \quad f_{xx}(0, 0) = 0$$

$$\frac{\partial^2}{\partial x \partial y} f(x, y) = abe^{ax} \cos by \quad \Rightarrow \quad f_{xy}(0, 0) = ab$$

$$\frac{\partial^2}{\partial y^2} f(x, y) = -b^2 e^{ax} \sin by \quad \Rightarrow \quad f_{yy}(0, 0) = 0$$

Scanned by CamScanner

Similarly, $\frac{\partial^3}{\partial x^3} f(0,0) = 0, \ \frac{\partial^3}{\partial x^2 \partial y} f(0,0) = 0$

$$\frac{\partial^3}{\partial x \partial y^2} f(0,0) = -ab^2, \ \frac{\partial^3}{\partial y^3} f(0,0) = -b^3$$

$$\Rightarrow \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0,0) = by, \ \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0,0) = 2abxy$$

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^3 f(0,0) = -3ab^2xy^2 - b^3y^3$$

$\therefore$ By Maclaurins formula we have

$$c^{ax} \sin by = 0 + by + \frac{2ab\,xy}{2!} + \frac{-(3ab^2xy^2 + b^3y^3)}{3!} + \ldots\ldots$$

2. Expand $f(x, y) = \frac{y^2}{x^3}$ upto second degree in $(x - 1)$, $(y + 1)$.

Solution : Here $f(1, -1) = 1$, $f_x = -\frac{3y^2}{x^4}$

$$\Rightarrow f_x(1, -1) = -3$$

$$f_y = \frac{2y}{x^3} \Rightarrow f_y(1, -1) = -2,$$

$$f_{xx} = \frac{12y^2}{x^5} \Rightarrow f_{xx}(1, -1) = 12,$$

$$f_{xy} = -\frac{6y}{x^4} \Rightarrow f_{xy}(1, -1) = 6,$$

$$f_{yy} = \frac{2}{x^3} \Rightarrow f_{yy}(1, -1) = 2.$$

Scanned by CamScanner

$\therefore$ By Taylor's formuls (5), we get

$$f(x, y) = f(1, -1) + \frac{1}{1!}\left((x-1)\frac{\partial}{\partial x} + (y+1)\frac{\partial}{\partial y}\right) f(1, -1)$$

$$+\frac{1}{2!}\left((x-1)\frac{\partial}{\partial x} + (y+1)\frac{\partial}{\partial y}\right)^2 f(1,-1) + ...$$

$$= 1 + \frac{1}{1!}((x-1)(-3) + (y+1)(-2))$$

$$+ \frac{1}{2!}((x-1)^2 \cdot 12 + (x-1)(y+1)6 + (y+1)^2 \cdot 2) + ...$$

$$= 1 - [3(x-1) + 2(y+1)]$$
$$+ [6(x-1)^2 + 3(x-1)(y+1) + (y+1)^2] + ... +$$

3. Prove that $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin xy + xe^x + y}{x\cos y + \sin 2y} = +\dfrac{3}{5}$,

if $(x, y) \to (0, 0)$ along st.line $y = 2x$.

Solution : $f(x, y) = \sin xy + xe^x - y$, $g(x, y) = x \cos y + \sin 2y$

$\Rightarrow$ $f_x = y \cos xy + e^x(x + 1)$, $f_y = x \cos (xy) + 1)$

$g_x = \cos y$, $g_y = -x \sin y + 2 \cos 2y$

$\Rightarrow$ $f_x(0, 0) = 1$, $f_y(0, 0) = +1$, $g_x(0, 0) = 1$, $g_y(0, 0) = 2$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{f(x,y)}{g(x,y)} = \frac{1+2(+1)}{1+2(2)} = +\frac{3}{5}$$

Scanned by CamScanner

Singular points :

Ordinary point and singular point :

Let $f(x, y) = 0$ be a continuous function such that f_x, f_y exist (1) A point on the curve $f(x, y) = 0$ at which f_x and f_y do not vanish simultaneously is called ordinary point, (2) a point of this curve at which f_x and f_y both vanish simultaneously is called a singular point, In Chapter 1 we have studied a point of inflexion which is a singular point. There are other kinds of singularities, we shall define them here.

Multiple point :

A point on a curve is called a multiple point of r^{th} order if r branches pass through the point.

If $r = 2$, then multiple point is called a double point.

If $r = 3$, then multiple point is called a triple point and so on.

It is obvious that r tangents, one to each branch can be drawn at a multiple point of r^{th} order. Also at such point, each of these r tangents cuts its own branch in one point and each of the other branches in two points. So $(r + 1)$ points altogether coincide at a multiple points of r^{th} order and so such tangent cuts the curve of n^{th} degree in $n - r - 1$ points. Thus two tangents can be drawn to the curve, one to each branch at a double point on the curve. These tangents may be real and distinct, real and coincident or imaginary. So double point is of three types.

Node : A double point on the curve at which two real and distinct tangents can be drawn is called a node.

Cusp : A double point on the curve at which two real and coincident tangents can be drawn is called a cusp.

Species of cusps : A cusp is called single or double according as the curve lies entirely on one side of the normal or on both sides. Also a cusp might be of the first species (or keratoid like horns) or the second species (or Ramphoid like a beak) according

Scanned by CamScanner

as the two branches lie on opposite sides or on the same side of the tangent. So we have the following types of cusps.

Single cusp of the first species in Fig. 1 single cusp of the second species in Fig. 2 double cusp of the first species in Fig. 4

Double cusp with change of species which is called osculinflexion in Fig. 5.

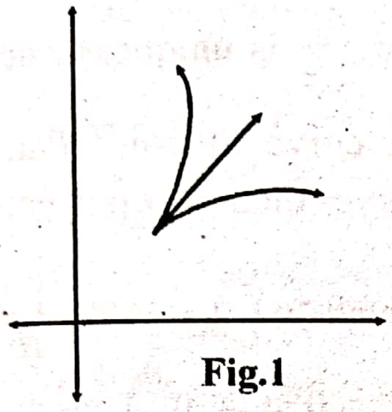


Fig.1

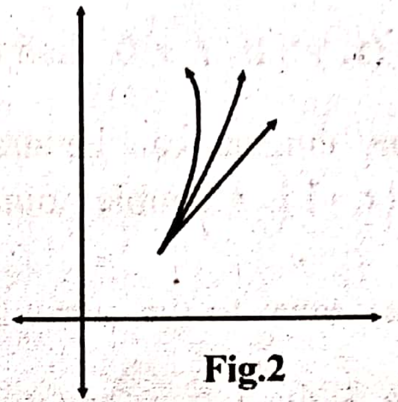


Fig.2

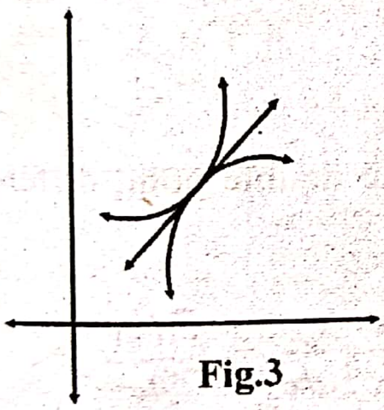


Fig.3

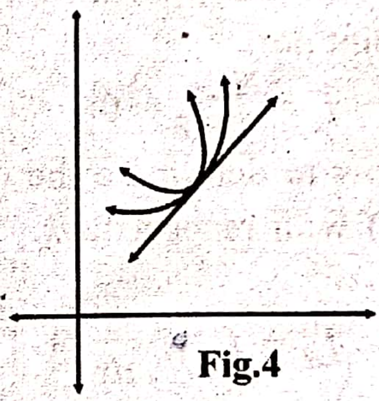


Fig.4

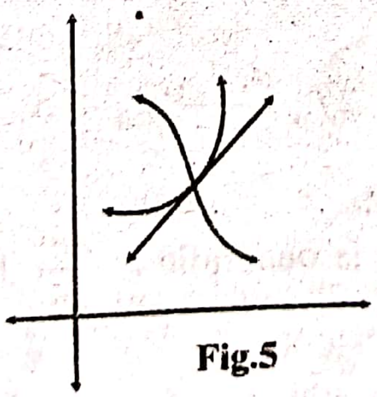


Fig.5

(3) **Isolated point :** A double point on the curve at which two imaginary tangents can be drawn is said to be isolated point.

Scanned by CamScanner

4.4 A necessary condition for the existance of a double point on the curve $f(x, y) = 0$.

Since $f(x, y) = 0$ ...(1)

$$\Rightarrow \quad f_x + f_y \cdot \frac{dy}{dx} = 0 \quad ...(2)$$

Equation (2) gives the slope of tangent to the curve at point $P(x, y)$. If $f_y \neq 0$, then the value of $\frac{dy}{dx}$ is unique. Hence only one tangent can be drawn to the curve at point $P(x, y)$. If $P(x, y)$ is a double point on the curve, then two tangents must be drawn to the curve at P for which two value of $\frac{dy}{dx}$ must be derived from equation (2). If $f_x = 0$, $f_y = 0$. Thus the conditions that point $P(x, y)$ on the curve $f(x, y) = 0$ is the double point are

$$f_x = 0, f_y = 0 \quad ...(3)$$

Now to determine the nature of the double point differentiate (2) with respect to x.

$$f_{xx} + f_{xy} \cdot \frac{dy}{dx} + \left(f_{yx} + f_{yy}\frac{dy}{dx}\right)\frac{dy}{dx} + f_y \frac{d^2y}{dx^2} \quad ...(4)$$

But $f_x = f_y = 0$

$$\therefore \quad f_{yy}\left(\frac{dy}{dx}\right)^2 + 2f_{xy}\frac{dy}{dx} + f_{xx} = 0 \quad ...(5)$$

This equation (4) is quadratic in $\frac{dy}{dx}$. Hence the nature of the roots i.e. $\frac{dv}{dx}$ of equation (4) gives the nature of double point. For equation (4), the discriminat $\Delta = 4\,((f_{xy})^2 - f_{xx} f_{yy}) = 4(s^2 - rt)$ say

Case (1) : If $s^2 - rt > 0$, then two values of $\frac{dy}{dx}$ are real and distinct. Hence two real and distinct tangents can be drawn to the curve at $P(x, y)$. Hence $P(x, y)$ is a node.

Case (2) : If $s^2 - rt > 0$, then two values of $\frac{dy}{dx}$ are real and coincident. Hence two coincident tangents can be drawn to the curve at $P(x, y)$. Hence $P(x, y)$ is a cusp.

Case (3) : If $s^2 - rt > 0$, then two values of $\frac{dy}{dx}$ are complex (imaginary). So two tangents to the curve at $P(x, y)$ are imaginary i.e. no tangent can be drawn to the curve. Hence $P(x, y)$ is conjugate point.

Thus in general, a double point will be a node cusp or conjugate point according as

$$(f_{xy})^2 - f_{xx} f_{yy} > \,, = \,, < 0$$

The curve has two branches at cusp. If for one branch, sign of y'' is positive and for second branch, sign of y'' is negative, then branches lie on opposite sides of the tangent. So the cusp is a cusp of the first species but for both branches, sign of y'' is same then both the branches lie on the same side of the tangent. So the cusp is the cusp of second species.

Note : For a curve $f(x, y) = 0$, to find the equations of tangents to the curve at origin, equate the lowest degree terms in $f(x, y)$ to zero say $\phi(x, y) = 0$ which gives the combined equation of its tangents at origin.

4.5 Illustrations :

Find the double point of the followings :

1. $x^3 + x^2 - 4y^2 = 0.$

Solution : Here $f(x, y) = x^3 + x^2 - 4y^2$

$\Rightarrow \quad f_x = 3x^2 + 2x, \quad f_y = -8y$

$\Rightarrow \quad f_{xx} = 6x + 2, \quad f_{yy} = -8$ and $f_{xy} = 0$

3. $(x-1)^2 + (x-1)^{5/2} - y + 3 = 0$

Solution : Here $(x-1)^2 + (x-1)^{5/2} - y + 3 = 0$ ...(1)

$\therefore$ Shift the origin the point (1, 3) and consider the current co-ordinates X, Y with respect to new co-ordinate axes.

$\therefore$ The equation (1) reduces to

$$F(X, Y) \equiv X^2 + X^{5/2} - Y = 0 \text{ or } (Y - X^2)^2 = X^5$$

$\therefore$ The tangents at origin are $X = 0$, $Y = 0$

$\therefore$ (0, 0) is the cusp of curve (2).

$\therefore$ (3, 1) is the cusp of given curve (1).

$$F_x = 2X + \frac{5}{2}X^{3/2},\ F_Y = 2(Y - X^2)$$

$$F_{xx} = 2 + \frac{15}{4}X^{1/2},\ F_{YY} = 0,\ F_{XY} = 0$$

$\therefore$ For a double point $F_X = 0$, $F_Y = 0$

$$\Rightarrow X\left(2 + \frac{5}{2}\sqrt{X}\right) = 0 \text{ or } Y = X^2 = 0$$

$$\Rightarrow X = 0,\ Y = 0$$

$$r = 2,\ s = 0,\ t = 0$$

$$\Rightarrow s^2 - rt = 0$$

$\therefore$ (0, 0) is the cusp of curve (2).

$\therefore$ (3, 1) is the cusp of given curve (1).

Scanned by CamScanner

4. $xy^2 - (x + y)^2 = 0$

Solution : Here $f(x, y) = xy^2 - (x + y)^2$

$\Rightarrow f_x = y^2 - 2(x + y), \quad f_y = 2xy - 2(x + y)$

For double points $f_x = 0, \quad f_y = 0$

$\Rightarrow y^2 = 2(x + y), \quad 2xy = 2(x + y)$

$\Rightarrow y^2 = 2xy$

$\Rightarrow y = 0 \quad \text{or} \quad y = 2x$

If $y = 0$, then $x = 0$

If $y = 2x$, then $4x^2 = 6x$

$\Rightarrow x = 0 \quad \text{or} \quad x = \frac{3}{2}$

If $x = 0, \ y = 0$, If $x = \frac{3}{2}, \ y = 3$

$\therefore$ The points are $(0, 0)$, $\left(\frac{3}{2}, 3\right)$.

But $(0, 0)$ lies on the curve while $\left(\frac{3}{2}, 3\right)$ does not lie on the curve.

$\therefore$ $(0, 0)$ is the double point.

Now tangents at origin are $(x + y) = 0$,

$\therefore$ The tangents at origin to the curve are real and coincident.

$\therefore$ $(0, 0)$ is the cusp

or $f_{xx}(0, 0) = -2, \quad f_{yy}(0, 0) = -2, \quad f_{xy}(0, 0) = -2$

$\therefore$ $(f_{xy})^2 - f_{xx} f_{yy} = (-2)^2 - (-2)(-2) = 0$

$\therefore$ $(0, 0)$ is the cusp.

Now $xy^2 - (x + y)^2 = 0 \Rightarrow y = \dfrac{x}{\pm\sqrt{x-1}}$

Scanned by CamScanner

If $x \in R^+$, $y \in R$, and $x \in R^-$, $y \notin R$.

$$\text{Now} \quad y = \frac{x}{\pm\sqrt{x-1}} = -x\left(1 \pm \sqrt{x}\right)^{-1}$$

$$= -x\left(1 \pm \sqrt{x}\right) \qquad \text{(taking } x \text{ very small)}$$

$$= -x \pm x^{3/2}$$

$$\Rightarrow \quad y' = -1 \mp \frac{3}{2} x^{1/2}$$

$$\Rightarrow \quad y'' = \mp \frac{3}{4} x^{-1/2}$$

$\therefore$ The branches of the curve lie on the opposite sides of the tangent at origin. Hence origin is the cusp of first species.

5. $\mathbf{x^3 + y^3 - 3x^2 - 3xy + 3x + 3y - 1 = 0.}$

Solution : Here $f(x, y) = x^3 + y^3 - 3x^2 - 3xy + 3x + 3y - 1$

$\Rightarrow$ $f_x = 3x^2 - 6x - 3y + 3$

$f_y = 3y^2 - 3x + 3$

For double points $f_x = 0$, $f_y = 0$

$\therefore$ $(x - 1)^2 - y = 0$, $x - 1 - y^2 = 0$

$\Rightarrow$ $y^4 - y = 0 \Rightarrow y(y^3 - 1) = 0$

$\Rightarrow$ $y = 0$, $y = 1$

If $y = 0$, $x = 1$, If $y = 1$, $x = 2$

$\therefore$ The points are $(1, 0)$ and $(2, 1)$

From which $(1, 0)$ lies on the curve but $(2, 1)$ does not lie on the curve. Shift the origin at point $(1, 0)$ and let the con-current co-ordinate be (X, Y). Then the equation

$$f(x, y) = (x - 1)^3 + y^3 - 3y(x - 1) = 0$$

reduces to $X^3 + Y^3 - 3XY = 0$.

$\therefore$ The tangents at new origin are $X = 0$, $Y = 0$

Scanned by CamScanner

4.7 Radius of curvature of a curve $y = f(x)$:

Since $y = f(x)$

$$\Rightarrow \quad \frac{dy}{dx} = f'(x) = \tan\psi$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \sec^2\psi \cdot \frac{d\psi}{dx}$$

$$= \sec^2\psi \cdot \frac{d\psi}{ds} \cdot \frac{ds}{dx}$$

$$= (1 + \tan^2\psi) \frac{d\psi}{ds} \cdot \frac{ds}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \left(1 + \left(\frac{dy}{dx}\right)^2\right)^1 \cdot \frac{1}{\varrho}\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}}$$

Scanned by CamScanner

$$\therefore \qquad \varrho = \pm \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left\{1+y'^2\right\}^{\frac{3}{2}}}{y''}$$

Radius of curvature of a curve $x = f(t), \quad y = g(t)$.

Since $\qquad x = f(t), \quad y = g(t)$

$$\Rightarrow \qquad \dot{x} = \frac{dx}{dt} = f'(t), \quad \dot{y} = \frac{dy}{dt} = g'(t)$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} \qquad \ldots(1)$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx}$$

$$= \frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right) \cdot \frac{1}{x} = \frac{\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x}}{\dot{x}^2} \cdot \frac{1}{\dot{x}}$$

$$= \frac{\dot{x}\,\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^3} \qquad \ldots(2)$$

$$\therefore \qquad \varrho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

$$= \left\{1 + \frac{\dot{y}^2}{\dot{x}^2}\right\}^{\frac{3}{2}} + \frac{\dot{y}\,\ddot{x} - \ddot{x}\,\dot{y}}{\dot{x}^3}$$

$$= \frac{\left\{\dot{x}^2 + \dot{y}^2\right\}^{\frac{3}{2}}}{\dot{x}\,\ddot{y} - \ddot{x}\,\dot{y}}$$

Scanned by CamScanner

6. **Find the radius of curvature of the curve**

$$\theta = a^{-1}\sqrt{r^2 - a^2} - \cos^{-1}\left(\frac{a}{r}\right).$$

Solution : Here $\theta = \frac{1}{a}\sqrt{r^2 - a^2} - \cos^{-1}\left(\frac{a}{r}\right)$. ...(1)

$\therefore$ Differentiating (1) w. r. to r

$$\frac{d\theta}{dr} = \frac{1}{a}\cdot\frac{2r}{2\sqrt{r^2 - a^2}} + \frac{1}{\sqrt{1 - \frac{a^2}{r^2}}}\left(-\frac{a}{r^2}\right)$$

$$= \frac{\sqrt{r^2 - a^2}}{ar}$$

$$\tan\phi = r\frac{d\theta}{dr} = r\frac{\sqrt{r^2 - a^2}}{ar} = \frac{\sqrt{r^2 - a^2}}{a}$$

$$\Rightarrow \quad \sin\phi = \frac{\sqrt{r^2 - a^2}}{\sqrt{r^2 - a^2 + a^2}} = \frac{\sqrt{r^2 - a^2}}{r}$$

$$\Rightarrow \quad p = r\sin\phi = \sqrt{r^2 - a^2}$$

$$\Rightarrow \quad \frac{dp}{dr} = \frac{r}{\sqrt{r^2 - a^2}}$$

$$\Rightarrow \quad \varrho = r\cdot\frac{dr}{dp} = r\cdot\frac{\sqrt{r^2 - a^2}}{r} = \sqrt{r^2 - a^2}$$

Scanned by CamScanner